# Homotopy Sequence for Fundamental Groups

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# Introduction

Let $f : X \to Y$ be a morphism between to topological spaces. A classical theorem in homotopy theory says that if $f$ is a fibration [**WH**][Chapter IV, Corollary 8.6] and $F := f^{-1}(y)$ is a fibre of some $y \in Y$, $x \in F$, then we have a long exact sequence of pointed sets

$$\cdots \to \pi_2^{\text{top}}(Y, y) \to \pi_1^{\text{top}}(F, x) \to \pi_1^{\text{top}}(X, x) \to \pi_1^{\text{top}}(Y, y) \to \pi_0^{\text{top}}(F, x) \to \pi_0^{\text{top}}(X, x).$$

If the fibre $F$ is connected then $\pi_0^{\text{top}}(F, x) = \{x\}$, so we get an exact sequence of groups

$$\cdots \to \pi_2^{\text{top}}(Y, y) \to \pi_1^{\text{top}}(F, x) \to \pi_1^{\text{top}}(X, x) \to \pi_1^{\text{top}}(Y, y) \to 1.$$

The above sequence is called the homotopy sequence (for $f$ and $y$).

If $\{ X_\alpha \mid \alpha \in I \}$ is a family of pathwise-connected topological spaces, $x_\alpha \in X_\alpha$, then the canonical morphism

$$\pi_n^{\text{top}}(\prod_{\alpha \in I} X_\alpha, \prod x_\alpha) \to \prod_{\alpha \in I} \pi_n^{\text{top}}(X_\alpha, x_\alpha)$$

is an isomorphism for all $n \in \mathbb{N}$.

In [**SGA1**][Exposé V] Grothendieck constructed for any locally noetherian connected scheme $X$ and any geometric point $x \to X$ a profinite group $\pi_1^{\text{ét}}(X, x)$ which is the analogue of the topological fundamental group. In fact if $X$ is a smooth connected complete scheme over $\mathbb{C}$ then $\pi_1^{\text{ét}}(X, x)$ is just the profinite completion of $\pi_1^{\text{top}}(X^{an}, x)$. If $f : X \to S$ is a separable proper morphism with geometrically connected fibres between locally noetherian connected schemes, $x \to X$ is a geometric point with image $s \to S$, Grothendieck shows in [**SGA1**, Exposé X, Corollaire 1.4] that one has a homotopy exact sequence for the étale fundamental group:

$$\pi_1^{\text{ét}}(\bar{X}_s, x) \to \pi_1^{\text{ét}}(X, x) \to \pi_1^{\text{ét}}(S, s) \to 1.$$

A similar case is that one can take $X, Y$ to be two locally noetherian connected $k$-schemes with $k = \bar{k}$ and suppose $Y$ is proper over $k$, so if $K$ is an algebraically closed field containing $k$ and if we take a $K$-point $z = (x, y) : \mathrm{Spec}\,(K) \to X \times_k Y$, then we get a canonical morphism of topological groups

$$\pi_1^{\text{ét}}(X \times_k Y, z) \to \pi_1^{\text{ét}}(X, x) \times \pi_1^{\text{ét}}(Y, y).$$

Again Grothendieck shows in [**SGA1**, Exposé X, Corollaire 1.7] that the canonical homomorphism is an isomorphism. This is called the Künneth formula for the étale fundamental group. If we take $K$ to be $k$ itself then the Künneth formula is a direct consequence of the homotopy exact sequence. If $k \subsetneqq K$ then one has to apply the following base change theorem [**SGA1**, Exposé X, Corollaire 1.8] to reduce to the case when $k = K$. The base change theorem states that the canonical map

$$\pi_1^{\text{ét}}(X \times_k K, x \times_k K) \to \pi_1^{\text{ét}}(X, x)$$

is an isomorphism between topological groups. Note that the base change theorem can be thought of as a special case of the Künneth formula by taking $Y = \text{Spec}\,(K)$, but it is not a corollary for logical reasons.

Let $X$ be a reduced connected scheme over a field $k$, $x \in X(k)$ be a rational point. If we set $N(X, x)$ to be the category whose objects consist of triples $(P, G, p)$ (where $P$ is an FPQC $G$-torsor over $X$, $G$ is a finite group scheme, $p \in P(k)$ is a $k$-rational point lying over $x$), whose morphisms are morphisms of $X$-schemes which are compatible with the group action. M.Nori proved in [**Nori**, Part I, Chapter II, Proposition 2] that the projective limit $\varprojlim_{N(X,x)} G$ exists in the category of $k$-group schemes (in the projective system we associate to each index $(P, G, p)$ the group $G$). Then he defined the fundamental group $\pi^N(X, x)$ to be the projective limit $\varprojlim_{N(X,x)} G$ which is called Nori's fundamental group nowadays. If $X$ is in addition proper over $k$ and if $k$ is perfect, Nori gave in [**Nori**, Part I, Chapter I] a Tannakian description of his fundamental group: he defined $\pi^N(X, x)$ to be the Tannakian group of the neutral Tannakian category of $\text{Ess}(X)$ (the essentially finite vector bundles on $X$) with the fibre functor $x^* : V \mapsto V|_x$, and he showed that this definition is the same as the one defined by the projective limit. If $X$ is smooth instead of proper and $k$ is a perfect field of characteristic $p > 0$, H.Esnault and A.Hogadi gave another Tannakian description of Nori's fundamental group in [**EH**][Section 3 and 4]. They defined $\pi^N(X, x)$ to be the Tannakian category of finite generalized stratified bundles with the fibre functor $x^* : (V_i, \sigma_i, i \geq 0) \mapsto V_0|_x$, and they showed that this definition coincide with the one defined via projective limit.

If $X$ is a smooth connected scheme over a field $k$ with a rational point $x \in X(k)$, we can consider the category of $O_X$-coherent $D_{X/k}$-modules which we will denote by $\mathsf{Mod}_c(D_{X/k})$. Now let $\omega_x$ be the functor $\mathsf{Mod}_c(D_{X/k}) \to \text{Vec}_k$ sending any $O_X$-coherent $D_{X/k}$-module $M$ to $M|_x$. One can check that the category $\mathsf{Mod}_c(D_{X/k})$ together with $\omega_x$ is a neutral Tannakian category, then we define its Tannakian group $\pi^{\text{alg}}(X, x)$ to be the algebraic fundamental group of $(X, x)$.

Nori's fundamental group and the algebraic fundamental group are all in some sense generalizations of Grothendieck's étale fundamental group. If $X$ is a connected reduced scheme over an algebraically closed field $k$, $x \in X(k)$, then $\pi_1^{\text{ét}}(X, x)$ is the $k$-points of the pro-étale quotient of $\pi^N(X, x)$. In particular if $k$ has characteristic 0, then $\pi_1^{\text{ét}}(X, x)$ is just the $k$-points of $\pi^N(X, x)$. If $X$ is a smooth connected scheme over a field $k$, $x \in X(k)$, then $\pi_1^{\text{ét}}(X, x)$ is the $k$-points of the profinite completion of $\pi^{\text{alg}}(X, x)$. So it is a natural question to ask how about the homotopy sequence and Künneth formula for Nori's fundamental group and the algebraic fundamental group. The main theme of this thesis is to answer this question.

In Chapter 1 we collect some basic definitions, results which will be used in later discussions. To start with, we introduce in §1 our major technical tool–Tannakian category.

This is a very beautiful categorical characterization of the finite representations of an affine group scheme. Most of the materials can be find in [**De2**] and [**De3**]. Since this theory has been well developed and has very nice references, we only give a very brief introduction here to make our thesis self-contained. Then we come to Nori's fundamental group. This is our major player. We start with the general definition of Nori's fundamental group, where we emphasize the existence of the projective limit $\varprojlim_{N(X,x)} G$. In fact this is not very hard to prove, the main point is that $N(X, x)$ is a filtered category if $X$ is reduced and connected. We then come to Nori's functor. Nori observed that giving a pointed torsor (torsor with a fixed rational point) is the same as giving a functor satisfying certain conditions. In §2.2 we reformulate Nori's idea in a more categorical language–the category of pointed torsors $N(X, x)$ is equivalent to the category of Nori's functors. The construction of the equivalence is very natural, but some details are not very easy to check. Next we give a sketch for the Tannakian description of Nori's fundamental group both in the proper case and in the smooth case. The main tool for the construction is of course Nori's functor. The general philosophy is that a torsor $\pi : P \to X$ over a proper base $X$ is somehow determined by the corresponding vector bundle $\pi_*O_P$ while for a smooth base $X$ one needs a sequence of bundles to identify it.

In Chapter 2, we discuss the homotopy sequence for Nori's fundamental group. Part of this work is based on the earlier work of H.Esnault, P.H.Hai , E.Viehweg in [**EHV**]. In [**EHV**][Section 2] H.Esnault, P.H.Hai , E.Viehweg, give a counterexample which shows that homotopy sequence of Nori's fundamental group is not always exact even for $X \to S$ projective smooth and $S$ projective smooth as well. And then they give a necessary and sufficient condition for the exactness of the homotopy sequence of Nori's fundamental group under the assumption that $S$ is a proper $k$-scheme. But unfortunately there is a gap in the argument for the necessary and sufficient condition. In this chapter, our first goal is to reformulate some similar conditions to make everything work. These works are contained in Theorem 1.0.23 and Theorem 2.0.4, where we correct the mistake, improve the arguments and make the wonderful ideas hidden in that article right and clean. The upshot is that in Theorem 1.0.23 we don't have to assume $S$ to be proper, so the result applies to the general definition of Nori's fundamental group.

Then we make two applications of Theorem 1.0.23 and Theorem 2.0.4. We first apply the criterion to show that the homotopy sequence for the étale quotient of Nori's fundamental group is exact. The argument is independent of Grothendieck's theory of the étale fundamental group which was developed in [**SGA1**, Exposé X], so it can be seen as a new proof of the homotopy exact sequence for the étale fundamental group (in the language of Nori's fundamental group). In [**MS**][Theorem 2.3] V.B.Mehta and S.Subramanian proved that Künneth formula holds for Nori's fundamental group if both $X$ and $Y$ are proper $k$-schemes. In §2, we apply Theorem 2.0.4 to give a neat proof for the Künneth formula of the local quotient of Nori's fundamental group. This can be thought of as a new proof of [**MS**][Proposition 2.1] which is the key point for the proof of [**MS**][Theorem 2.3].

In the end of this chapter, we give a counterexample to show that [**MS**][Theorem 2.3] does not work if $X$ or $Y$ is not proper, where we take $X = \mathbb{A}^1_k$ and $Y = E$ to be a supersingular elliptic curve and $k$ to be an algebraically closed field of characteristic 2. This also provides another counter example to show the failure of the exactness of the homotopy sequence for Nori's fundamental group (in the split case).

In Chapter 3, we proved that homotopy sequence is exact for the algebraic fundamental group in characteristic 0. The proof is very tricky. Our major tool is the criterion for the exactness of a sequence of affine group schemes provided by the properties of the corresponding functors. There are three conditions (a), (b), (c) in the criterion, we have to check them one by one. Among them (a) and (b) are relatively easy to check while condition (c) is very difficult. We do not prove (c) directly, instead we prove (c) for a special case: "the generic geometric point" (Chapter 3, §2), this part arises from a cleaning work of the letter [**E**], the main idea is from that letter. Then we come back and say that if the homotopy sequence is exact in this special case then it is exact in general. Unfortunately, in the way we reduce our problem to the special case we used some transcendental method.

Although we strongly believe that the homotopy sequence is also exact in characteristic $p$, we could not prove it at this moment. But in the end of this chapter we obtained the exactness for a special case–the Künneth formula. This is an easy consequence of Phùng Hô Hai's work on 0-th Gauss-Manin for stratified bundles [**Hai**].

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## Contents

CHAPTER 1

# Preliminaries

## 1. Tannakian Formalism

The notion of a neutral Tannakian category can be thought of as a linearization of the notion of a Galois category which is developped in [**SGA1**][Exposé V]. It gives a characterization of the category of finite representations of an affine group scheme. In this section we will recall briefly some basics about neutral Tannakian categories which will serve as the main technical tool in our following discussions. For more details one can find in [**De2**] and [**De3**].

DEFINITION 1.0.1. Let $\mathfrak{C}$ be a category and

$$\otimes : \mathfrak{C} \times \mathfrak{C} \to \mathfrak{C}$$
$$(X, Y) \mapsto X \otimes Y$$

a functor which satisfies

(1) "A": an associativity constraint for $(\mathfrak{C}, \otimes)$ is a functorial isomorphism

$$\phi_{X,Y,Z} : X \otimes (Y \otimes Z) \xrightarrow{\cong} (X \otimes Y) \otimes Z$$

which satisfies the commutative diagram for all the objets $X, Y, Z, T \in \mathfrak{C}$

$$\begin{tikzcd} X \otimes (Y \otimes (Z \otimes T)) \arrow[r, "\phi"] \arrow[d, "1\otimes\phi"'] & (X \otimes Y) \otimes (Z \otimes T) \arrow[r, "\phi"] & ((X \otimes Y) \otimes Z) \otimes T \\ X \otimes ((Y \otimes Z) \otimes T) \arrow[rr, "\phi"] & & (X \otimes (Y \otimes Z)) \otimes T \arrow[u, "\phi\otimes 1"'] \end{tikzcd}.$$

(2) "C": a commutativity contraint for $(\mathfrak{C}, \otimes)$ is a funtorial isomorphism

$$\varphi_{X,Y} : X \otimes Y \xrightarrow{\cong} Y \otimes X$$

which satisfies

$$\varphi_{Y,X} \circ \varphi_{X,Y} = id_{X \otimes Y}.$$

Moreover $\phi$ and $\varphi$ are compatible in the following sense: the diagram

$$\begin{tikzcd} X \otimes (Y \otimes Z) \arrow[r, "\phi"] \arrow[d, "1\otimes\varphi"'] & (X \otimes Y) \otimes Z \arrow[r, "\varphi"] & Z \otimes (X \otimes Y) \arrow[d, "\phi"] \\ X \otimes (Z \otimes Y) \arrow[r, "\phi"] & (X \otimes Z) \otimes Y \arrow[r, "\varphi\otimes 1"] & (Z \otimes X) \otimes Y \end{tikzcd}$$

is commutative.

(3) "U": the unit: a pair $(U, u)$ consists of an object $U \in \mathfrak{C}$ and an isomorphism $u : U \to U \otimes U$ such that the functor $X \mapsto U \otimes X$ induces an equivalence of categories.

The category $\mathfrak{C}$ which satisfies the conditions "A", "C", "U" ("ACU" for short) above is called a $\otimes-$category (tensor category).

Lemma 1.0.2. *Let $(U, u)$ be the unit object of a tensor category $\mathfrak{C}$. Then*

*(1) There is a unique functorial isomorphism*

$$l_X : X \cong U \otimes X$$

*which satisfies*

*(a) $l_U = u$;*

*(b) The diagrams*

$$\begin{tikzcd} X \otimes Y \arrow[r, "l"] \arrow[d, equal] & U \otimes (X \otimes Y) \arrow[d, "\phi"] \\ X \otimes Y \arrow[r, "l \otimes 1"] & (U \otimes X) \otimes Y \end{tikzcd} \qquad \begin{tikzcd} X \otimes Y \arrow[r, "l \otimes 1"] \arrow[d, "1 \otimes l"'] & (U \otimes) X \otimes Y \arrow[d, "\varphi \otimes 1"] \\ X \otimes Y \arrow[r, "\phi"] & (U \otimes X) \otimes Y \end{tikzcd}$$

*are commutative.*

*(2) If $(U', u')$ is another unit of $\mathfrak{C}$, one has a unique isomorphism $t : U \to U'$ such that*

$$\begin{tikzcd} U \arrow[r, "u"] \arrow[d, "t"'] & U \otimes U \arrow[d, "t \otimes t"] \\ U' \arrow[r, "u'"] & U' \otimes U' \end{tikzcd}$$

*is a commutative diagram.*

Definition 1.0.3. A tensor category $\mathfrak{C}$ is called rigid if

(1) for $X, Y \in \mathfrak{C}$ the functor

$$\underline{\mathrm{Hom}}(X, Y) : T \to \mathrm{Hom}(T \otimes X, Y)$$

is representable;

(2) any $X \in \mathfrak{C}$ is reflexive, i.e. the canonical morphism

$$\underline{\mathrm{Hom}}(X, U) \otimes X \to U$$

which corresponds to the identity automorphism of $\underline{\mathrm{Hom}}(X, U)$ induces an isomorphism

$$X \to \underline{\mathrm{Hom}}(\underline{\mathrm{Hom}}(X, U), U);$$

(3) for all $X_1, X_2, Y_1, Y_2 \in \mathfrak{C}$ the canonical morphism

$$\underline{\mathrm{Hom}}(X_1, Y_1) \otimes \underline{\mathrm{Hom}}(X_2, Y_2) \to \underline{\mathrm{Hom}}(X_1 \otimes X_2, Y_1 \otimes Y_2),$$

induced from

$$\begin{aligned}&(\underline{\mathrm{Hom}}(X_1,Y_1)\otimes\underline{\mathrm{Hom}}(X_2,Y_2))\otimes(X_1\otimes X_2)\\ \cong&(\underline{\mathrm{Hom}}(X_1,Y_1)\otimes X_1)\otimes(\underline{\mathrm{Hom}}(X_2,Y_2)\otimes X_2)\xrightarrow{\mathrm{ev}_1\otimes\mathrm{ev}_2}Y_1\otimes Y_2\end{aligned}$$

(where $\mathrm{ev}_i$ corresponds to the identity automorphism of $\underline{\mathrm{Hom}}(X_i,Y_i)$ $(i=1,2)$) is an isomorphism.

Definition 1.0.4. A tensor category $\mathfrak{C}$ is called an abelian tensor category if it is equipped with a structure of an abelian category which makes $\otimes$ a bi-additive functor.

Remark 1.0.5. Let $\mathfrak{C}$ be an abelian tensor category. If $R:=\mathrm{End}(U)$ is a commutative ring, $\mathrm{Hom}(X,Y)$ is naturally an $R-$modules for all the objets $X,Y\in\mathfrak{C}$.

Definition 1.0.6. Let $k$ be a field. An abelian tensor category $\mathfrak{C}$ is called $k$-linear if there is an isomorphism $\mathrm{End}(U)\cong k$, where $U$ is the unit of $\mathfrak{C}$.

Definition 1.0.7. A tensor functor between two tensor categories $(\mathfrak{C},\otimes)$ and $(\mathfrak{C}',\otimes')$ is a pair $(F,c)$ which consists of a functor $F$ and functorial isomorphisms $c_{X,Y}:F(X\otimes Y)\cong F(X)\otimes F(Y)$ with the following properties:

(1) A: for all $X,Y,Z\in\mathfrak{C}$, the diagram

$$\begin{tikzcd}
F(X)\otimes(F(Y)\otimes F(Z)) \arrow[r, "id\otimes c"] \arrow[d, "\phi'"'] & F(X)\otimes F(Y\otimes Z) \arrow[r, "c"] & F(X\otimes(Y\otimes Z)) \arrow[d, "\phi"] \\
(F(X)\otimes F(Y))\otimes F(Z) \arrow[r, "c\otimes id"] & F(X\otimes Y)\otimes F(Z) \arrow[r, "c"] & F((X\otimes Y)\otimes Z)
\end{tikzcd}$$

is commutative;

(2) C: for all $X,Y\in\mathfrak{C}$, the diagram

$$\begin{tikzcd}
F(X\otimes Y) \arrow[r, "c"] \arrow[d, "F(\varphi)"'] & F(X)\otimes F(Y) \arrow[d, "\varphi'"] \\
F(Y\otimes X) \arrow[r, "c"] & F(Y)\otimes F(X)
\end{tikzcd}$$

is commutative;

(3) U: $(F(U),F(u))=(U',u')$, où $(U,u)$ is the unit of $\mathfrak{C}$ and $(U',u')$ is the unit of $\mathfrak{C}'$.

Proposition 1.0.8. *Let $(F,c)$ be a tensor functor between two rigid tensor categories $\mathfrak{C}$ and $\mathfrak{C}'$. The canonical morphism $F(\underline{\mathrm{Hom}}(X,Y))\to\underline{\mathrm{Hom}}(F(X),F(Y))$ is an isomorphism for all $X,Y\in\mathfrak{C}$.*

Definition 1.0.9. A tensor functor $(F,c):\mathfrak{C}\to\mathfrak{C}'$ is called a tensor equivalence if $F$ is an equivalence of categories. Let $(G,d):\mathfrak{C}\to\mathfrak{C}'$ be another tensor functor. A morphism of tensor functors $(F,c)\to(G,d)$ is a morphism of functors $\lambda:F\to G$ which makes the following diagram commutative:

$$\begin{tikzcd} \otimes_{i\in I}F(X_i) \arrow[r, "c"] \arrow[d, "\lambda_{X_i}"'] & F(\otimes_{i\in I}X_i) \arrow[d, "\lambda_{(\otimes_{i\in I}X_i)}"] \\ \otimes_{i\in I}G(X_i) \arrow[r, "d"] & G(\otimes_{i\in I}X_i) \end{tikzcd} ,$$

for $I=\{1,2\}$ or $I=\emptyset$, $X_i\in\mathfrak{C}$.

PROPOSITION 1.0.10. *Let $(F,c):\mathfrak{C}\to\mathfrak{C}'$ be a tensor equivalence. There is a tensor functor $(G,d):\mathfrak{C}'\to\mathfrak{C}$ such that $G\circ F\cong id_{\mathfrak{C}}$ and $F\circ G\cong id_{\mathfrak{C}'}$, where all the isomorphisms of functors are tensor isomorphisms.*

DEFINITION 1.0.11. Let $k$ be a field. A neutral Tannakian category is a rigid $k$-linear abelian tensor category $\mathfrak{C}$ equipped with a $k$-linear exact faithful tensor functor

$$\omega:\mathfrak{C}\to \mathrm{Vec}_k,$$

where $\mathrm{Vec}_k$ is the category of finite dimensional $k$-vector spaces.

DEFINITION 1.0.12. Let $\mathfrak{C}$ be a neutral Tannakian category with the fibre functor $\omega:\mathfrak{C}\to\mathrm{Vec}_k$. We denote by $\mathrm{Aut}^{\otimes}(\omega)$ the functor from the category of $k$-schemes to the category of groups which associates to each $k$-scheme $X$ the group of tensor automorphisms of the following functor

$$\mathfrak{C}\longrightarrow \mathsf{Mod}(O_X)$$
$$T\mapsto \omega(T)\otimes_k O_X.$$

THEOREM 1.0.13. *Let $\mathfrak{C}$ be a neutral Tannakian category with the fibre functor $\omega:\mathfrak{C}\to$ $\mathrm{Vec}_k$. Then*

*(1)* $\mathrm{Aut}^{\otimes}(\omega)$ *is representable by an affine group scheme $G$ over $k$;*
*(2) There is a $k$-linear tensor equivalence $h:\mathfrak{C}\to\mathsf{Rep}_k(G)$ such that $F\circ h=\omega$, where $F$ is the forgetful functor $\mathsf{Rep}_k(G)\to\mathrm{Vec}_k$.*

THEOREM 1.0.14. *Let $G$, $G'$ be affine group schemes over a field $k$, $\omega$, $\omega'$ be the forgetful functors $\mathsf{Rep}_k(G)\to\mathrm{Vec}_k$, $\mathsf{Rep}_k(G')\to\mathrm{Vec}_k$ respectively. For any tensor functor $T:$ $\mathsf{Rep}_k(G')\to\mathsf{Rep}_k(G)$ such that $\omega\circ T\cong\omega'$ (tensor $k$-linear isomorphic), there is a unique morphism of $k$-group schemes $f:G\to G'$ such that the functor $\omega^f:\mathsf{Rep}_k(G')\to\mathsf{Rep}_k(G)$ induced from $f$ is tensor $k$-linear isomorphic to $T$.*

DEFINITION 1.0.15. ([**De2**][6.16]) Let $\mathfrak{C}$ be a rigid abelian tensor category. For an object $X\in\mathfrak{C}$, the tensor category generated by $X$ is the full sub-category of $\mathfrak{C}$ which consists of the sub-quotients of the direct sums of $\{X^{\otimes n}|n\in\mathbb{Z}\}$, where $X^{\otimes n}$ for $n$ negative stands for $\underline{\mathrm{Hom}}(X^{\otimes(-n)},U)$.

THEOREM 1.0.16. *Let $G$ be an affine $k$-group scheme.*

*(1) $G$ is finite over $k$ if and only if there exists an object $X\in\mathsf{Rep}_k(G)$ such that each object of $\mathsf{Rep}_k(G)$ is isomorphic to a sub-quotient of $X^n$ ($n-$th direct sum of $X$) for $n\geq 0$.*

(2) *$G$ is algebraic if and only if there is an object $X \in \mathsf{Rep}_k(G)$ such that $X$ is a tensor generator of $\mathsf{Rep}_k(G)$.*

DEFINITION 1.0.17. Let $\mathfrak{C}$ be a rigid $k$-linear abelian tensor category, $S$ a $k$-scheme. A fibre functor of $\mathfrak{C}$ over $S$ is a $k-$linear exact tensor functor $\omega$ from $\mathfrak{C}$ to the category of quasi-coherent sheaves on $S$.

THEOREM 1.0.18. *Let $\mathfrak{C}$ be a rigid $k$-linear abelian tensor category, $S = \operatorname{Spec}(B)$ a $k$-scheme. Then any two fibre functors of $\mathfrak{C}$ over $S$ are locally isomorphic in the FPQC-topology, i.e. there exists $S' = \operatorname{Spec}(B')$ on $S$, faithfully flat, such that all fibre functors $\omega_1$ and $\omega_2$ becomes isomorphic after an extension of scalars from $S$ to $S'$.*

THEOREM 1.0.19. ([**EPS**][Appendix Theorem A.1]) *Let $L \xrightarrow{q} G \xrightarrow{p} A$ be a sequence of homomorphisms of affine group schemes over a field $k$. It induces a sequence of functors:*

$$\mathsf{Rep}_k(A) \xrightarrow{p^*} \mathsf{Rep}_k(G) \xrightarrow{q^*} \mathsf{Rep}_k(L),$$

*where $\mathsf{Rep}_k(-)$ denotes the category of finite dimensional representations of $-$ over $k$. Then we have*

(1) *The group homomorphism $p : G \to A$ is faithfully flat if and only if $p^*\mathsf{Rep}_k(A)$ is a full subcategory of $\mathsf{Rep}_k(G)$ and closed under taking subquotients.*
(2) *The group homomorphism $q : L \to G$ is a closed immersion if and only if any object of $\mathsf{Rep}_k(L)$ is a subquotient of an object of the form $q^*(V)$ for some $V \in \mathsf{Rep}_k(G)$.*
(3) *Assume that $q$ is a closed immersion and that $p$ is faithfully flat. Then the sequence $L \xrightarrow{q} G \xrightarrow{p} A$ is exact if and only if the following conditions are fulfilled:*
    (a) *For an object $V \in \mathsf{Rep}_k(G)$, $q^*V \in \mathsf{Rep}_k(L)$ is trivial if and only if $V \cong p^*U$ for some $U \in \mathsf{Rep}_k(A)$*
    (b) *Let $W_0$ be the maximal trivial subobject of $q^*V$ in $\mathsf{Rep}_k(L)$. Then there exists $V_0 \subseteq V$ in $\mathsf{Rep}_k(G)$, such that $q^*V_0 \cong W_0$.*
    (c) *Any $W$ in $\mathsf{Rep}_k(L)$ is embeddable in $q^*V$ for some $V \in \mathsf{Rep}_k(G)$.*

## 2. Nori's Fundamental Group

Nori's fundamental group scheme is in some sense a generalization of Grothendieck's étale fundamental group. If $X$ is a connected reduced locally noetherian scheme over an algebraically closed field $k$ which admits a rational point, then the étale fundamental group is just the $k$-points of the pro-étale quotient of Nori's fundamental group. In this section will follow Nori's influential paper [**Nori**] and also include some recent development [**EH**]. We first give the most general notion of Nori's fundamental group for a connected reduced scheme over a field. Then we introduce a very powerful tool–Nori's functor–using which we can get the Tannakian descriptions of Nori's fundamental group for $X$ proper and $X$ smooth respectively.

### 2.1. The General Definition.

DEFINITION 2.1.1. Let $X$ be a scheme over a field $k$, $x \in X(k)$. We denote by $N(X, x)$ the category whose objects consist of triples $(P, G, q)$, where $P$ is an FPQC $G$-torsor over $X$, $G$ is a finite $k$-group scheme, $q \in P(k)$ is a $k$-rational point lying over $x$, whose morphisms between two objects $(P, G, q)$ and $(P', G', q')$ are pairs $(\phi, h)$, where $h : G \to G'$ is a homomorphism of $k$-group schemes and $\phi : P \to P'$ is an $X$-scheme morphism sending $q$ to $q'$ which is also compatible with the group actions. Note that $N(X, x)$ is equivalent to a small category.

DEFINITION 2.1.2. Let $I$ be a category which is equivalent to a small category. We say $I$ is filtered if it satisfies the following:

(1) for any $i, j \in I$ there exist an object $k$ and two morphisms $k \to i$, $k \to j$ in $I$;

(2) for any two arrows $u, v : j \to i$ there exists an arrow $w : k \to j$ such that $u \circ w = v \circ w$ in $\mathrm{Hom}(k, i)$.

If $I$ admits a final object, then the above is equivalent to the following: for any two morphisms $u : j \to i$ and $v : k \to i$ in $I$, there are two morphisms $a : l \to j$ and $b : l \to k$ such that $u \circ a = v \circ b$.

PROPOSITION 2.1.3. [**Nori**][Chapter II, Lemma 1] *If $X$ is a reduced connected locally noetherian scheme over a field $k$, $x \in X(k)$, then $N(X, x)$ is a filtered category.*

PROOF. It is enough to see that for any two morphisms

$$(\phi_i, h_i) : (P_i, G_i, p_i) \to (Q, H, q) \in N(X, x)$$

where $i = 1, 2$, the triple $(P_1 \times_Q P_2, G_1 \times_H G_2, p_1 \times_q p_2)$ is an object in our category $N(X, x)$.

First note that the triple $(P_1 \times_X P_2, G_1 \times_k G_2, p_1 \times_k p_2)$ is in $N(X, x)$ and that one has two closed imbeddings:

$$P_1 \times_Q P_2 \hookrightarrow P_1 \times_X P_2$$

and

$$G_1 \times_G G_2 \hookrightarrow G_1 \times_k G_2.$$

Furthermore, the action of $G_1 \times_k G_2$ on $P_1 \times_X P_2$ induces an action of $G_1 \times_G G_2$ on $P_1 \times_Q P_2$. In fact one can check easily that the following morphism

$$(P_1 \times_Q P_2) \times_k (G_1 \times_G G_2) \to (P_1 \times_Q P_2) \times_X (P_1 \times_Q P_2)$$

which is induced by the isomorphism

$$(P_1 \times_X P_2) \times_k (G_1 \times_k G_2) \to (P_1 \times_X P_2) \times_X (P_1 \times_X P_2)$$

is an isomorphism itself. Now let $Y$ be the quotient of $P_1 \times_Q P_2$ by $G_1 \times_G G_2$,

$$q : P_1 \times_Q P_2 \to Y$$

be the quotient map. Then there is a canonical finite morphism of schemes $i : Y \to X$ (because $X$ is invariant under the action of $G_1 \times_G G_2$). Consider the following commutative diagram:

$$\begin{tikzcd}
(P_1 \times_Q P_2) \times_k (G_1 \times_G G_2) \arrow[r, "\cong"] \arrow[d, "q \circ pr_1"'] & (P_1 \times_Q P_2) \times_X (P_1 \times_Q P_2) \arrow[d, "q \times q"] \\
Y \arrow[r, "\Delta"] & Y \times_X Y
\end{tikzcd}$$

Since $q$ is finite faithfully flat, the vertical arrows in the above diagram are all finite and faithfully flat, so $\Delta$ is also faithfully flat. But $\Delta$ is a closed immersion, thus it must be an isomorphism. Hence the finite morphism $i : Y \to X$ is a monomorphism in the category of schemes. Thus it has to be a closed immersion. Now look at the following diagram

$$\begin{tikzcd} P_1 \times_Q P_2 \arrow[r, hook] \arrow[d, two heads, "q"] & P_1 \times_X P_2 \arrow[d, two heads] \\ Y \arrow[r, "i"] & X \end{tikzcd}.$$

Since $P_1 \times_Q P_2$ is the fibre of the neutral element of $G$ under the following map

$$P_1 \times_X P_2 \xrightarrow{(\phi_1 \times \phi_2)} Q \times_X Q \xrightarrow{\cong} Q \times_k G \xrightarrow{pr_2} G,$$

so it must be both open and closed in $P_1 \times_X P_2$. But the map $P_1 \times_X P_2 \to X$ is finite flat, so $Y$, as the image of $P_1 \times_Q P_2$ under $P_1 \times_X P_2 \to X$ is both open and closed. As $X$ is connected and reduced we have $Y \to X$ is an isomorphism. Now we have $q : P_1 \times_Q P_2 \to Y = X$ is finite flat and the map

$$(P_1 \times_Q P_2) \times_k (G_1 \times_G G_2) \to (P_1 \times_Q P_2) \times_X (P_1 \times_Q P_2)$$

is an isomorphism, so the triple

$$(P_1 \times_Q P_2, G_1 \times_H G_2, p_1 \times_q p_2) \in N(X, x).$$

$\square$

Lemma 2.1.4. *Let $I$ be a filtered category, $X$ be a scheme. Let $\mathrm{Aff}(X)$ be the category of affine schemes over $X$ (i.e. the category of affine morphisms to $X$) and $F : I \to \mathrm{Aff}(X)$ be a functor. Then the projective limit*

$$\varprojlim_{i \in I} F(i)$$

*exists in the category of affine schemes over $X$.*

Remark. This lemma is a very standard fact in scheme theory, the proof is quite easy, we will leave it to the reader.

Theorem 2.1.5. [**Nori**][Chapter II, Proposition 2] *Let $X$ be a reduced connected locally noetherian scheme over a field $k$, $x \in X(k)$. Then there exists a triple $(\widetilde{X_x}, \pi^N(X, x), \tilde{x})$, where $\pi^N(X, x)$ is a profinite $k$-group scheme, $\widetilde{X_x}$ is a $\pi^N(X, x)$-torsor in FPQC topology over $X$, $\tilde{x} \in \widetilde{X_x}(k)$ is a rational point lying above $x$, which satisfies for any $(P, G, q) \in N(X, x)$ there exists a unique morphism*

$$(\phi, h) : (\widetilde{X_x}, \pi^N(X, x), \tilde{x}) \to (P, G, q),$$

*where $h : \pi^N(X, x) \to G$ is homomorphism of $k$-group schemes and $\phi : \widetilde{X_x} \to P$ is a morphism of $X$-schemes sending $\tilde{x}$ to $q$ which is also compatible with the group actions. The group scheme $\pi^N(X, x)$ is usually called the Nori's fundamental group.*

PROOF. We have two functors

$$F_X : N(X,x) \to \mathrm{Aff}(X), \qquad (P,G,q) \mapsto P$$

and

$$\pi_X : N(X,x) \to \mathrm{Aff}(\mathrm{Spec}\,(k)), \qquad (P,G,q) \mapsto G.$$

By the lemma above we have two projective limits

$$\widetilde{X_x} := \varprojlim_{i \in N(X,x)} F_X(i) \qquad \text{and} \qquad \pi^N(X,x) := \varprojlim_{i \in N(X,x)} \pi_X(i).$$

$\widetilde{X_x}$ is an affine scheme over $X$ which admits a rational $k$-point $\tilde{x}$ obtained by the universality of the projective limit. $\pi^N(X,x)$ is an affine scheme over $k$, but it also carries a structure of an affine group scheme over $k$. This follows from the following simple formula:

$$\varprojlim_{i \in N(X,x)} \pi_X(i) \times_k \varprojlim_{i \in N(X,x)} \pi_X(i) \cong \varprojlim_{i \in N(X,x)} (\pi_X(i) \times_k \pi_X(i)).$$

This canonical isomorphism defines for us the multiplication of $\varprojlim_{i \in N(X,x)} \pi_X(i)$:

$$\varprojlim_{i \in N(X,x)} \pi_X(i) \times_k \varprojlim_{i \in N(X,x)} \pi_X(i) \to \varprojlim_{i \in N(X,x)} \pi_X(i).$$

Furthermore with this group scheme structure $\pi^N(X,x)$ becomes a projective limit of these $\pi_X(i), i \in N(X,x)$ in the category of affine group schemes.

Now the triple $(\widetilde{X_x}, \pi^N(X,x), \tilde{x})$ has the property that for any $i := (P,G,q) \in N(X,x)$ there is a map

$$(\phi_i, h_i) : (\widetilde{X_x}, \pi^N(X,x), \tilde{x}) \to (P,G,q)$$

defined by seeing $i = (P,G,q) \in N(X,x)$ as an index. To see that the map $(\phi_i, h_i)$ is unique we suppose there is another map

$$(\phi, h) : (\widetilde{X_x}, \pi^N(X,x), \tilde{x}) \to (P,G,q).$$

Considering the construction of $\pi^N(X,x)$ in terms of Hopf-algebras, it is clear that there is an index $j := (P', G', q') \in N(X,x)$ such that $h$ factors the canonical map

$$h_j : \pi^N(X,x) \to G'.$$

Thus we get a commutative diagram

$$\begin{tikzcd} & (\widetilde{X_x}, \pi^N(X,x), \tilde{x}) \arrow[dl, "(\phi_j,h_j)"'] \arrow[dr, "(\phi,h)"] & \\ (P',G',q') \arrow[rr, "(\varphi,g)"] & & (P,G,q) \end{tikzcd} .$$

But by the very definition of a projective limit, we know that $(\varphi, g) \circ (\phi_j, h_j) = (\phi_i, h_i)$. Thus $(\phi_i, h_i) = (\phi, h)$. This completes the proof. $\square$

PROPOSITION 2.1.6. [**Nori**][Chapter II, Proposition 4] *Let $X$ be a connected reduced locally notherian scheme over a field $k$, $x, y \in X(k)$. Then $\pi^N(X,x)$ is an inner twist* *of $\pi^N(X,y)$ and they are isomorphic (non-canonically) after pulling back to the algebraic closure of $k$.*

PROPOSITION 2.1.7. [**Nori**][Chapter II, Proposition 5]*Let $X$ be a connected locally noetherian scheme separable over a field $k$, $x \in X(k)$. If $k \subset l$ is a separable algebraic extension, then the canonical map*

$$\pi^N(X \times_k l, x \times_k l) \to \pi^N(X,x) \times_k l$$

*is an isomorphism of $l$-group schemes.*

PROPOSITION 2.1.8. [**Nori**][Chapter II, Proposition 6]*Let $f : X \to S$ be a proper separable surjective morphism between two connected reduced locally noetherian schemes over a field $k$ with geometrically connected fibres, $s \in S(k)$, $x \in X(k)$ such that $f(x) = s$. Then the induced map*

$$\pi^N(X,x) \to \pi^N(S,s)$$

*is surjective.*

PROOF. We only have to show that for any $(P,G,q) \in N(S,s)$ which corresponds to a surjection $\pi^N(S,s) \twoheadrightarrow G$ remains a surjection after composing with the canonical map $\pi^N(X,x) \to \pi^N(S,s)$.

Let $H$ be the image of the composition $\pi^N(X,x) \to \pi^N(S,s) \to G$. This map gives us via Theorem 2.1.5 a morphism

$$(P',H,q') \to f^*(P,G,q) \in N(X,x).$$

Let $V$ be the 0-th direct image of the structure map $P' \to X$. Then by the conditions we have imposed on $f$, there is a vector bundle $W$ on $S$ such that $f^*W \cong V$. One checks readily that $W$ is a sheaf of $O_S$-algebras and it carries an action from $H$ which makes it a $H$-torsor. Thus if we write $P_1 := \mathrm{Spec}\,(W)$ then we get an object $(P_1,H,q_1) \in N(S,s)$ and a morphism

$$(P_1,H,q_1) \to (P,G,q) \in N(S,s)$$

extending the inclusion $H \subseteq G$. So by Theorem 2.1.5 we get a morphism $\pi^N(S,s) \to H$ which factors the surjection $\pi^N(S,s) \twoheadrightarrow G$. This implies $H \subseteq G$ is actually an isomorphism. $\square$

## 2.2. Nori's Functor.

DEFINITION 2.2.1. Let $X$ be a scheme over a field $k$, $x : S \to X$ be a $k$-morphism. We will use $P(X,x)$ to denote the category whose objects consist of triples $(P,G,q)$, where $P$ is an FPQC $G$-torsor over $X$, $G$ is an affine $k$-group scheme, $p : S \to P$ is a $k$-morphism which after composing with the projection $P \to X$ is $x$, whose morphisms between two objects $(P,G,q)$ and $(P',G',q')$ are pairs $(\phi,h)$, where $h : G \to G'$ is a homomorphism of $k$-group schemes and $\phi : P \to P'$ is an $X$-scheme morphism sending $q$ to $q'$ which is also compatible with the group actions.

DEFINITION 2.2.2. Let $X$ and $S$ be schemes over a field $k$, $x : S \to X$ be a $k$-morphism. We will use $F(X,x)$ to denote the category whose objects are triples $(\mathrm{Fib},G,\psi)$, where $G$

is an affine $k$-group scheme,

$$\mathrm{Fib} : \mathsf{Rep}_k(G) \to \mathrm{Coh}(X)$$

is a fibre functor (faithful exact $k$-linear tensor functor), $\psi$ is an isomorphism of tensor functors between $x^* \circ \mathrm{Fib} : \mathsf{Rep}_k(G) \to \mathrm{Coh}(S)$ and the composition of the forgetful functor with the canonical pull-back functor, as is indicated in the following 2-commutative diagram

$$\begin{tikzcd} \mathsf{Rep}_k(G) \arrow[r] \arrow[d] & \mathrm{Coh}(X) \arrow[d, "x^*"] \\ \mathrm{Vec}_k \arrow[r] & \mathrm{Coh}(S) \end{tikzcd} ,$$

whose morphisms between two objects $(\mathrm{Fib}, G, \psi)$ and $(\mathrm{Fib}', G', \psi')$ are pairs $(\phi, h)$, where $h : G \to G'$ is morphisms of $k$-group schemes, $\phi$ is an isomorphism of tensor functors making the diagram

$$\begin{tikzcd} \mathsf{Rep}_k(G') \arrow[dr, "\mathrm{Fib}'"] \arrow[dd, "h^*"'] & \\ & \mathrm{Coh}(X) \\ \mathsf{Rep}_k(G) \arrow[ur, "\mathrm{Fib}"'] & \end{tikzcd}$$

2-commutative and is compatible with $\psi$ and $\psi'$ in the obvious way.

THEOREM 2.2.3. [**Nori**][Chapter I, Proposition 2.9] *There is a natural equivalence between $P(X, x)$ and $F(X, x)$.*

PROOF. We will set up two functors

$$\Delta : P(X, x) \to F(X, x) \qquad \text{and} \qquad \nabla : F(X, x) \to P(X, x)$$

and then prove that they are quasi-inverse to each other.

Given $(P, G, q) \in P(X, x)$, by lemma 1.31 below we know that for any $V \in \mathsf{Rep}_k(G)$ the scheme $P \times^G \mathbb{A}_V$ ($\mathbb{A}_V := \mathrm{Spec}\,(\mathrm{Sym}_k(V))$) is a vector bundle of rank $\dim_k V$, and this operation is functorial in $V$. So we have defined a functor

$$\mathrm{Fib} : \mathsf{Rep}_k(G) \to \mathrm{Coh}(X) \qquad \text{sending} \qquad V \mapsto P \times^G \mathbb{A}_V.$$

Now we pull back the vector bundle $P \times^G \mathbb{A}_V$ along $x : S \to X$. Then we get

$$\psi_V : S \times_X P \times^G \mathbb{A}_V \cong S \times_k G \times^G \mathbb{A}_V \cong S \times_k \mathbb{A}_V,$$

where the first isomorphism is given by the section $p : S \to P$ and the second isomorphism is canonical. This gives us the desired isomorphism of functors $\psi$ from $x^* \circ \mathrm{Fib}$ to the composition of the forgetful functor with the pull-back functor. If we have a morphism

$$(\phi, h) : (P, G, q) \to (P', G', q') \in P(X, x),$$

then for any $V \in \mathsf{Rep}_k(G)$, $\phi$ induces a morphism

$$P \times^G \mathbb{A}_V \to P' \times^G \mathbb{A}_V.$$

This gives the desired isomorphism of functors which makes the diagram

$$\begin{tikzcd} \mathsf{Rep}_k(G') \arrow[d, "h^*"'] \arrow[dr, "\mathrm{Fib}'"] & \\ & \mathrm{Coh}(X) \\ \mathsf{Rep}_k(G) \arrow[ur, "\mathrm{Fib}"'] & \end{tikzcd}$$

commutative. And one can check that this isomorphism is also compatible with $\psi$ and $\psi'$. So we get the functor $\Delta$.

Now suppose we have a triple $(\mathrm{Fib}, G, \psi) \in F(X, x)$. Let $k[G] \in \mathsf{Rep}_k(G)$ be the right regular representation. Then $\mathrm{Fib}(k[G])$ is a coherent sheaf on $X$. By [**De2**][2.8] $\mathrm{Fib}(k[G])$ is a vector bundle. Since Fib is tensor functor we get a ring structure on $\mathrm{Fib}(k[G])$ by defining the multiplication as

$$\mathrm{Fib}(k[G]) \otimes_{O_X} \mathrm{Fib}(k[G]) \to \mathrm{Fib}(k[G]) := \mathrm{Fib}(k[G] \otimes_k k[G] \to k[G]),$$

the unit as

$$\mathrm{Fib}(k) \to \mathrm{Fib}(k[G]) := \mathrm{Fib}(k \to k[G]).$$

Since $\mathrm{Fib}(k[G])$ is a vector bundle, $P := \mathrm{Spec}_{O_X}(\mathrm{Fib}(k[G]))$ is finite faithfully flat $X$-scheme. Because the right regular representation of $k[G]$ comes from the left translation of $G$ and the map

$$G_1 \times_k G_2 \xrightarrow{\cong} G_3 \times_k G_4 \qquad (x, y) \mapsto (x, xy)$$

is $G$-invariant (where $G_1 = G_2 = G_3 = G_4 = G$ as $k$-group schemes but $G_2$ is equipped with the trivial $G$-action while the others are equipped with the left translation), so we can apply $\mathrm{Fib}(-)$ to the corresponding map

$$k[G_3] \times_k k[G_4] \xrightarrow{\cong} k[G_1] \times_k k[G_2]$$

and get

$$\mathrm{Fib}(k[G_3]) \otimes_{O_X} \mathrm{Fib}(k[G_4]) \xrightarrow{\cong} \mathrm{Fib}(k[G_1]) \otimes_{O_X} \mathrm{Fib}(k[G_2]).$$

As $k[G_2]$ is equipped with the trivial $G$-action, $\mathrm{Spec}_{O_X}(\mathrm{Fib}(k[G_2])) = G \times_k X$. So if we apply $\mathrm{Spec}_{O_X}(-)$ to the above isomorphism, we will get an isomorphism of $X$-schemes

$$P \times_k G \xrightarrow{\cong} P \times_X P.$$

One checks readily that this map composing with the second projection $P \times_X P \to P$ gives an action of $G$ on $P$. So $P$ equipped with this action is an FPQC $G$-torsor over $X$. As $\psi$ is an isomorphism of tensor functors so it induces an isomorphism

$$x^*\mathrm{Fib}(k[G]) \cong k[G] \otimes_k O_S$$

as sheaves of $O_S$-algebras. Taking spectrum on both sides we get an isomorphism

$$G \times_k S \xrightarrow{\cong} P \times_X S.$$

The identity point of $G \times_k S$ gives us an $S$-point $p : S \to P$ lifting $x : S \to X$. Now suppose we have a morphism

$$(\phi, h) : (\mathrm{Fib}, G, \psi) \to (\mathrm{Fib}', G', \psi') \in F(X, x).$$

We have canonical morphisms of $O_X$-algebras

$$\mathrm{Fib}'(k[G']) \cong \mathrm{Fib}(k[G']) \to \mathrm{Fib}(k[G]),$$

where the first arrow is induced by $\phi$ and the second $k[G']$ is equipped with the action of $G$ induced by $h$. These morphisms will induce a morphism of $X$-schemes $\varphi : P \to P'$. It is very easy to see that $\varphi$ is compatible with the actions, and since $\phi$ is compatible with $\psi$ and $\psi'$ we also see that $\varphi$ sends $p \mapsto p'$. This gives the functor $\nabla$.

One can check that $\Delta$ and $\nabla$ are quasi-inverse to each other. So we have the equivalence of categories. $\square$

Remark. Let $(P, G, q) \in P(X, x)$, $(\mathrm{Fib}, G, \psi) := \Delta(P, G, q) \in F(X, x)$. The fibre functor Fib is usually called Nori's functor in the literature.

Proposition 2.2.4 (Definition-Proposition). *Let $X$ be a scheme, $G$ be a group scheme over $X$, $P$ be a right $G$-torsor in FPQC-topology, $F/X$ be an affine $X$-scheme with a $G$-action, then there exists an affine $X$-scheme $P \times^G F$ and a morphism of $X$-schemes $P \times F \to P \times^G F$ which can be regarded as the quotient of $P \times F$ (as FPQC-sheaves) under the $G$-action: $g \times (p, x) \mapsto (pg, g^{-1}x)$. This $P \times^G F$ is called the contracted product of $P$ and $F$.*

Proof. In the following we set $P_1 = P_2 = P$. Let $\phi : P_1 \times P_2 \times F \to P_1 \times P_2 \times F$ be the $P_1 \times P_2$-map sending $(p, pg, f)$ to $(p, pg, g^{-1}f)$. It is not hard to check $\phi$ satisfies the cocycle condition, thus the $P-$scheme $P \times F$ can be descent to an affine $X$-scheme $P \times^G F$ and we have a cartesian diagram:

$$\begin{tikzcd} P \times F \arrow[r, "\lambda"] \arrow[d] & P \times^G F \arrow[d] \\ P \arrow[r] & X \end{tikzcd} .$$

Now consider the following map

$$\rho : P \times P \times F \to P \times F \qquad \text{sending} \qquad (p, pg, f) \mapsto (pg, g^{-1}f).$$

If we regard the $X$-scheme $P \times F$ as the descent of the $P$-scheme

$$pr_2 : P \times P \times F \to P,$$

then it is easily seen that $\rho$ is compatible with the descent data on both sides. Thus $\rho$ descends to an $X$-scheme morphism $\varphi : P \times F \to P \times^G F$ making the following diagram:

$$\begin{tikzcd} P \times P \times F \arrow[r, "\rho"] \arrow[d, "pr_{13}"'] & P \times F \arrow[r, dotted, "pr_1"] \arrow[d, "\lambda"] & P \arrow[d, dotted] \\ P \times F \arrow[r, "\varphi"] & P \times^G F \arrow[r, dotted] & X \end{tikzcd}$$

commutative. But the diagram

$$\begin{tikzcd} P \times P \times F \arrow[r, "\rho"] \arrow[d, "pr_{13}"'] & P \times F \arrow[d, "\varphi"] \\ P \times F \arrow[r, "\varphi"] & P \times^G F \end{tikzcd}$$

is already commutative as one can see this by pulling the diagram back along $\lambda : P \times F \to P \times^G F$. This implies $\varphi = \lambda$ (because $\rho$ is FPQC). So the left square of the commutative diagram

$$\begin{tikzcd} P \times P \times F \arrow[r, "\rho"] \arrow[d, "pr_{13}"'] & P \times F \arrow[r, dotted, "pr_1"] \arrow[d, "\lambda"] & P \arrow[d, dotted] \\ P \times F \arrow[r, "\lambda"] & P \times^G F \arrow[r, dotted] & X \end{tikzcd}$$

is cartesian (because it is the case for the left square and the composition of the two squares). This implies $\lambda$ is the FPQC-quotient. □

Lemma 2.2.5. *Let $G$ be a group scheme over a field $k$, $P$ be a right $G$-torsor in FPQC-topology over an $k$-scheme $X$, $V \in \mathsf{Rep}_k(G)$, $\mathbb{A}_V = \mathrm{Spec}\,(\mathrm{Sym}_k(V))$, then $P \times^G \mathbb{A}_V$ is a $\dim_k V$ dimensional vector space over $X$.*

Proof. In the proof of the above proposition we set $F = \mathbb{A}_V \times_k X$, $G/X = G \times_k X$. If we know $\phi : P_1 \times P_2 \times F \to P_1 \times P_2 \times F$ is induced from an automorphism of $V \otimes_k O_{P_1 \times P_2}$ then we can descent $V \otimes_k O_P$ to a locally free $O_X-$module $\mathcal{E}$ and the symmetric $O_X$-algebra of it can be identified with $P \times^G \mathbb{A}_V$ in the category of $X$-schemes. Then by definition $P \times^G \mathbb{A}_V$ is a vector bundle of $\dim_k V$ over $X$. So, now our task is to prove $\phi$ is induced from an automorphism of the $k-$vector space $V$. Let $F_n$ be the $X-$scheme $\mathrm{Spec}(S_0 + S_1 + \cdots + S_n)$, where $S_i$ are the homogenous components of $\mathrm{Sym}_{O_X}(O_X \otimes_k V)$. Then for each $n$ we have natural closed immersion $i_n$ making the following diagram commutative:

$$\begin{tikzcd} P_1 \times P_2 \times F_n \arrow[r, "\phi_n"] \arrow[d, "i_n"'] & P_2 \times P_1 \times F_n \arrow[d, "i_n"] \\ P_1 \times P_2 \times F \arrow[r, "\phi"'] & P_2 \times P_1 \times F \end{tikzcd}$$

This tells us the automorphism of $\mathrm{Sym}_{O_X}(O_X \otimes_k V)$ induced by $\phi$ is a homogenous map, thus there exists an automorphism of $V \otimes_k O_{P_1 \times P_2}$ as $O_{P_1 \times P_2}-$modules which induces $\phi$. This concludes the proof. □

**2.3. Nori's Fundamental Group on a Proper Base.** In this subsection we assume $X$ is a proper reduced connected scheme over a perfect field $k$. All the results in this section are taken from [**Nori**][Chapter II, §2].

Definition 2.3.1. A vector bundle $V$ on $X$ is called finite if there are two polynomials $f(X), g(X) \in \mathbb{N}(X)$ (i.e. polynomials with non-negative integer coefficients) such that $f(X) \neq g(X)$ but $f(V) = g(V)$.

Lemma 2.3.2. *Let $V$ be a vector bundle on $X$, $S(V)$ be the set of all isomorphic classes of the indecomposable components of $V^{\otimes n}$ for $n \in \mathbb{N}$, then $V$ is finite if and only if $S(V)$ is a finite set.*

Remark. In the proof of this lemma one needs Krull-Remak-Schmidt theorem for coherent sheaves. So it is crucial that the base $X$ is proper. Note that if a vector bundle $V$ can be written as a direct sum $V = M \oplus N$ of coherent sheaves then $M$ and $N$ are automatically vector bundles, so we can apply the theorem here.

Corollary 2.3.3. *The category of finite vector bundles is stable under taking finite direct sums, direct summands, finite tensor products, duals. More specifically we have for any two vector bundles $V_1$ and $V_2$ on $X$*

*(1) $V_1, V_2$ is finite $\Rightarrow V_1 \oplus V_2$, $V_1 \otimes V_2$, $V_1^{\vee}$ are finite;*

*(2) $V_1 \oplus V_2$ is finite $\Rightarrow V_1$ is finite.*

Proof. The proofs of the items of (1) are quite similar, we take $V_1 \otimes V_2$ as an example. Since $S(V_1)$ and $S(V_2)$ are all finite, we can take all isomorphic classes of the indecomposable components of $E_1 \otimes E_2$ for all $E_1 \in S(V_1)$ and $E_2 \in S(V_2)$. This set is finite by Krull-Remak-Schmidt theorem. And one can see easily that this set is precisely $S(V_1 \oplus V_2)$.

For (2) one just has to observe that $S(V_1) \subseteq S(V_1 \oplus V_2)$. □

Lemma 2.3.4. *If $X$ is a proper smooth geometrically connected curve over $k$, then a finite vector bundle on $X$ is semistable of degree 0.*

Definition 2.3.5. Let $X$ be a proper connected reduced scheme over a perfect field $k$. A vector bundle $V$ on $X$ is called semi-stable of degree 0 if for any proper smooth geometrically connected curve and any morphism $f : Y \to X$ which is birational onto its image the pull-back $f^*V$ is semi-stable of degree 0. Since any complete curve is also projective so it makes sense to talk about semi-stable bundles on $X$ by just assuming $X$ proper (not projective).

Corollary 2.3.6. *A finite vector bundle on $X$ is always semi-stable of degree 0.*

Proposition 2.3.7. *The full subcategory $SS(X) \subseteq \mathrm{Coh}(X)$ of the category of coherent sheaves on $X$ which consists of semi-stable vector bunldes of degree 0 as objects is an abelian category.*

Proof. First note that the claim is true if $X$ is a proper smooth connected curve. So if $V \to W$ is an injective map of semi-stable vector bundles of degree 0 over $X$ then it is enough to show that both the kernel and cokernel are vector bundles. But each two closed points of $X$ are contained in a integral closed sub-scheme of dimension 1 in $X$, and each integral closed sub-scheme of dimension 1 is the image of some proper smooth connected curve $f : Y \to X$ with $f$ birational. So the map $V \to W$ has constant rank after restricting to each closed point. This tells us that the kernel and cokernel of $V \to W$ are all vector bundles. □

REMARK. This proposition is not true if $X$ is not proper. For example if $X$ is affine, then the only connected closed sub-schemes which are proper over $k$ are the closed points. This implies for any map $f : Y \to X$ with $Y$ a proper smooth connected curve, the image is just a closed point. So all vector bundles on $X$ are semi-stable of degree 0. But all the vector bundles certainly does not form an abelian category in general. So the properness is crucial here.

DEFINITION 2.3.8. We use $\mathrm{Ess}(X)$ to denote the full sub category of $SS(X)$ consists of objects which are subquotients of finite vector bundles. Objects in $\mathrm{Ess}(X)$ are called essentially finite vector bundles.

PROPOSITION 2.3.9. $\mathrm{Ess}(X)$ *is a $k$-linear abelian tensor category. If $X$ admits a $k$-rational point $x$ then* $\mathrm{End}(O_X) = k$ *and we have a canonical fibre functor $V \mapsto V|_x$, so then* $\mathrm{Ess}(X)$ *equipped with this fibre functor is a neutral Tannakian category.*

PROOF. $\mathrm{Ess}(X)$ is obviously an abelian category and is $k$-linear. The fact that it is also a tensor category is from Corollary 1.34. If $X$ admits a $k$-rational point then it is geometrically connected, so $\mathrm{End}(O_X) = \Gamma(X, O_X) = k$. So $\mathrm{Ess}(X)$ is a neutral Tannakian category. $\square$

LEMMA 2.3.10. *Let $x \in X(k)$ be a rational point. Let $P \to X$ be a $G$-torsor in FPQC-topology with $G$ a finite group scheme over $k$. By 1.2.2 there exists a fibre functor* $\mathsf{Rep}_k(G) \to \mathrm{Coh}(X)$. *This functor factors through the inclusion* $\mathrm{Ess}(X) \subseteq \mathrm{Coh}(X)$. *Conversely, if $V \in \mathrm{Ess}(X)$, then there exists a $G$-torsor $P \to X$ in FPQC-topology with $G$ a finite group scheme over $k$ such that $V$ is in the essential image of the induced fibre functor.*

REMARK. The first part of this lemma is basically from the fact that any finite representation is embeddable into a direct sum of regular representations. The second part of this lemma is from the fact that $V$ is a subquotient of finite vector bundles so the sub Tannakian category generated by this object is represented by a finite group scheme over $k$.

THEOREM 2.3.11. *Let $X$ be a proper reduced connected scheme over a perfect field $k$. $x \in X(k)$ be a $k$-rational point. $N(X, x)$ be the category of pointed torsors as in 1.2.1. Then for each object $i = (P, G, p) \in N(X, x)$ there exists a functor* $\mathsf{Rep}_k(G) \to \mathrm{Ess}(X)$ *and a 2-commutative diagram*

$$
\begin{tikzcd}
\mathsf{Rep}_k(G) \arrow[rr] \arrow[dr] & & \mathrm{Ess}(X) \arrow[dl] \\
& \mathrm{Vec}_k &
\end{tikzcd}
$$

*Since the correspondence is functorial (1.2.2) we can pass to the 2-direct limit and get a 2-commutative diagram*

$$\begin{tikzcd}
\mathsf{Rep}_k(\pi^N(X,x)) = \varinjlim_{i\in N(X,x)} \mathsf{Rep}_k(G_i) \arrow[rr] \arrow[dr] & & \mathrm{Ess}(X) \arrow[dl] \\
& \mathrm{Vec}_k &
\end{tikzcd}$$

*Then the horizontal arrow in the above diagram is a tensor equivalence. Furthermore, the above diagram corresponds via 1.2.2 to the universal triple $(\widetilde{X_x}, \pi^N(X,x), \widetilde{x})$ which we have discussed in 1.2.1.*

Remark. The above theorem follows easily from the above lemma and the universality of the triple $(\widetilde{X_x}, \pi^N(X,x), \widetilde{x})$. It gives a Tannakian description of $\pi^N(X,x)$. But unfortunately, there are two crucial points where properness is used. So it is a very interesting question to ask about the Tannakian description of $\pi^N(X,x)$ for non-proper $X$. In the next subsection we will discuss the Tannakian description of $\pi^N(X,x)$ for $X$ smooth which was developed by H.Esnault and A.Hogadi.

**2.4. Nori's Fundamental Group on a Smooth Base.** In this subsection we assume $X$ is a smooth connected scheme over a perfect field $k$ of characteristic $p > 0$. All the results in this section are taken from [**EH**].

Definition 2.4.1. For $i \in \mathbb{N}$, we have the relative Frobenius $\phi_i : X^{(i)} \to X^{(i+1)}$ starting with $X^{(0)} = X$. Let $t \in \mathbb{N}$. A $t$-stratified bundle $(E^{(i)}, \sigma^{(i)}, i \in \mathbb{N})$ consists of a sequence of vector bundles $E^{(i)}$ on $X^{(i)}$, and a sequence of $O_{X^{(i)}}$-isomorphisms

$$\sigma_i : E^{(i)} \to \phi_i^* E^{(i+1)}$$

for all $i \geq 1$ and for $i = 0$,

$$\sigma_0 : \phi_{(-t,0)}^* E^{(i)} \to \phi_{(-t,1)}^* E^{(i+1)}$$

is an $O_{X^{(-t)}}$-isomorphism, where $\phi_{(-t,0)} : X^{(-t)} \to X^{(0)}$ is the composition of relative Frobenius and similarly for $\phi_{(-t,1)}$.

Definition 2.4.2. Let $t \in \mathbb{N}$, $\mathrm{Strat}(X,t)$ be the category whose objects consist of $t$-stratified bundles, whose morphisms between two objects

$$\mathrm{Hom}((E^{(i)}, \sigma^{(i)}, i \in \mathbb{N}), (F^{(i)}, \tau^{(i)}, i \in \mathbb{N}))$$

is the set of morphisms $E^{(i)} \to F^{(i)}$ for $i \in \mathbb{N}$ which are compatible with $\sigma^{(i)}$ and $\tau^{(i)}$ in a natural way. Because of the faithful flatness of the relative Frobenius, one has a fully faithful imbedding $\mathrm{Strat}(X,t) \subseteq \mathrm{Strat}(X,t+1)$. Now taking the 2-direct limit in the category of categories, one gets a new category $\mathrm{Strat}(X,\infty) := \varinjlim_{t\geq 0} \mathrm{Strat}(X,t)$.

Construction 2.4.3. Let $x \in X(k)$, $(P, G, p) \in N(X, x)$. We want to construct out of $(P, G, p)$ a $k$-linear tensor functor $\eta_G$ making diagram

$$\begin{tikzcd} \mathsf{Rep}_k(G) \arrow[rr, "\eta_G"] \arrow[dr, "F_G"'] & & \mathrm{Strat}(X, \infty) \arrow[dl, "\omega_x"] \\ & \mathrm{Vec}_k & \end{tikzcd}$$

2-commutative, where $\omega_x$ is the functor sending $(E^{(i)}, \sigma^{(i)}, i \in \mathbb{N}) \mapsto E^{(0)}|_x$ and $F_G$ is the forgetful functor.

Now for each $X^{(i)}$ with $i \in \mathbb{N}$, by (1.2.2) we have a triple $(\mathrm{Fib}^{(i)}, G^{(i)}, \psi^{(i)}) \in F(X^{(i)}, x)$ corresponding to the triple $(P^{(i)}, G^{(i)}, p^{(i)}) \in N(X^{(i)}, x)$. Since $G^{(i)}$ is a finite group scheme over a perfect field $k$, we have the following decomposition:

$$1 \to G_0^{(i)} \to G^{(i)} \to G_{\text{ét}}^{(i)} \to 1,$$

and this exact sequence splits into a semi-direct product by a canonical section

$$\iota^{(i)} : G_{\text{ét}}^{(i)} = G_{\text{red}}^{(i)} \to G^{(i)}.$$

We will denote by $(\mathrm{Fib}_{\text{ét}}^{(i)}, G_{\text{ét}}^{(i)}, \psi_{\text{ét}}^{(i)})$ the triple corresponding to $(P_{\text{ét}}^{(i)}, G_{\text{ét}}^{(i)}, p^{(i)}) \in N(X^{(i)}, x)$.

For any $V \in \mathsf{Rep}_k(G)$, we define $E^{(0)} = \mathrm{Fib}^{(0)}(V)$. Because we have canonical isomorphisms $\phi_{(i,i+1)} : G_{\text{ét}}^{(i)} \cong G_{\text{ét}}^{(i+1)}$, $V$ can be regarded as an object in $\mathsf{Rep}_k(G_{\text{ét}}^{(i)})$ for all $i \in \mathbb{Z}$ via the section $\iota^{(0)} : G_{\text{ét}} \to G$. Hence we can definie for $i > 0$ $E^{(i)} = \mathrm{Fib}_{\text{ét}}^{(i)}(V)$. Since the diagram

$$\begin{tikzcd} P_{\text{ét}}^{(i)} \arrow[r] \arrow[d] & P_{\text{ét}}^{(i+1)} \arrow[d] \\ X^{(i)} \arrow[r] & X^{(i+1)} \end{tikzcd}$$

is cartesian, we get isomorphisms

$$\sigma^{(i)} : E^{(i)} \cong \phi_i^* E^{(i+1)}, \quad i \in \mathbb{N} \setminus \{0\}.$$

For $i = 0$ we know that there is a large enough nature number $t$ such that the relative Frobenius $G^{(-t)} \to G^{(0)} = G$ factors through $\iota : G_{\text{ét}} \to G$. Thus we get an isomorphism

$$\phi_{(-t,0)}^* E^{(0)} = \phi_{(-t,0)}^* \mathrm{Fib}^{(0)}(V) \cong \mathrm{Fib}_{\text{ét}}^{(-t)}(V) \cong \mathrm{Fib}_{\text{ét}}^{(0)}(V) \cong \mathrm{Fib}_{\text{ét}}^{(0)}(V) = \phi_{(-t,1)}^* E^{(1)}.$$

In this way we get a $t$-stratified bundle $(E^{(i)}, \sigma^{(i)}, i \in \mathbb{N})$ out of the triple $(P, G, p)$ and $V \in Rep_k(G)$. One can check that the association is functorial with respect to $V \in Rep_k(G)$. One can also check easily that we have the 2-commutative diagram as claimed above, so we have defined the functor $\eta_G$.

Note that our construction of $\eta_G$ is functorial with respect to $(P, G, p)$, so taking 2-direct limit in the 2-category of categories we get a functor $\eta$ and a 2-commutative diagram

$$\begin{tikzcd} \varinjlim_{i\in N(X,x)} \mathsf{Rep}_k(G_i) \arrow[rr, "\eta"] \arrow[dr, "F"'] & & \mathrm{Strat}(X,\infty) \arrow[dl, "\omega_x"] \\ & \mathrm{Vec}_k & \end{tikzcd} .$$

Theorem 2.4.4. (Esnault and Hogadi). *Let $X$ be a smooth connected scheme over a perfect field $k$ of positive characteristic with a rational point $x \in X$, $t \in \mathbb{N}$. Then the categories $\mathrm{Strat}(X,t)$, $\mathrm{Strat}(X,\infty)$ together with the fibre functor $(E^{(i)}, \sigma^{(i)}, i \in \mathbb{N}) \mapsto E^{(0)}|_x$ are neutral Tannakian categories. The functor $\eta$ defined above is a fully faithful imbedding of tensor $k-$linear categories. If we set $\mathrm{Strat}^{\mathrm{fin}}(X)$ to be the full subcategory of $\mathrm{Strat}(X,\infty)$ whose objects consists of those whose Tannakian groups are finite, then $\mathrm{Strat}^{\mathrm{fin}}(X)$ is the essential image of $\eta$. In other words $\eta$ induces an equivalence between $\varinjlim_{i\in N(X,x)} \mathsf{Rep}_k(G_i)$ and $\mathrm{Strat}^{\mathrm{fin}}(X)$.*

Remark. In [**EH**], they define $\pi^{\mathrm{alg},\infty}(X,x)$ to be the Tannakian group of $\mathrm{Strat}(X,\infty)$. Since $\mathrm{Strat}^{\mathrm{fin}}(X)$ as a subcateogry of $\mathrm{Strat}(X,\infty)$ is stable under subquotient, so we get a surjection $\pi^{\mathrm{alg},\infty}(X,x) \to \pi^N(X,x)$. So $\pi^N(X,x)$ is the profinite quotient of $\pi^{\mathrm{alg},\infty}(X,x)$. Now we get a Tannakian description of $\pi^N(X,x)$ under the smoothness assumption of $X/k$.

## 3. The Algebraic Fundamental Group

The algebraic fundamental group for a smooth geometrically connected scheme $X$ over a field $k$ is defined to be the Tannakian group of the neutral Tannakian category of $O_X$-coherent $D$-modules over $X/k$. But before we proceed any further we have to make sense of the tensor and internal Hom structures in the category of $O_X$-coherent $D$-modules. To do this we have to split our problem to two cases–characteristic 0 and characteristic $p$. In characteristic 0 the category of $D$-modules is equivalent to the category of integrable connections so we can do our job there. In characteristic $p$, by a theorem of Katz, the category of $O_X$-coherent $D$-modules is equivalent to the category of stratified bundles, so we will just concentrate on stratified bundles in characteristic $p$.

Definition 3.0.5. Let $f : X \to S$ be a morphism of schemes. Then we have the diagonal map $\Delta : X \hookrightarrow X \times_S X$. This gives us an exact sequence of $O_X$-modules:

$$0 \to \mathcal{I} \to \Delta^{-1}O_{X\times_S X} \to O_X \to 0.$$

We call the sheaf $\mathcal{P}^n_{X/S} := \Delta^{-1}O_{X\times_S X}/\mathcal{I}^{n+1}$ the sheaf of principal parts of order $n$ of $X/S$. We will regard $\mathcal{P}^n_{X/S}$ as an $O_X$-module using the first projection $pr_1 : X \times_S X \to X$. We call

$$\mathcal{D}iff^n_{X/S}(O_X) := \mathcal{H}om_{O_X}(\mathcal{P}^n_{X/S}, O_X)$$

the sheaf of differential operators of order $\leq n$ of $X/S$. $\mathcal{H}om_{O_X}(\mathcal{P}^n_{X/S}, O_X)$ can also be seen as a subsheaf of $\mathcal{E}nd_{f^{-1}O_S}(O_X)$ consisting of sections which factor through the map

$O_X \to \mathcal{P}^n_{X/S}$ defined by the second projection $pr_2 : X \times_S X \to X$. One has canonical inclusions $\mathcal{D}iff^n_{X/S}(O_X) \subseteq \mathcal{D}iff^{n+1}_{X/S}(O_X)$, one can take direct limit in the category of $O_X$-modules:

$$\mathcal{D}iff_{X/S}(O_X) := \varinjlim_{n\in\mathbb{N}} \mathcal{D}iff^n_{X/S}(O_X).$$

$\mathcal{D}iff_{X/S}(O_X)$ is an $O_X$-module which also carries an $O_X$-algebra structure (non-commutative).

Definition 3.0.6. Let $f : X \to S$ be a morphism of schemes, $E$ be sheaf of $O_X$-modules. A $D$-module structure on $E$ is a left $O_X$-algebra homomorphism

$$\nabla : \mathcal{D}iff_{X/S}(O_X) \to \mathcal{E}nd_{f^{-1}O_S}(E).$$

We will use $\mathsf{Mod}_c(D_{X/S})$ to denote the category of all $O_X$-coherent $D$-modules over $X/S$.

Lemma 3.0.7. *Let $f : X \to S$ be a morphism of schemes, $D \in \mathrm{Diff}^n_{X/S}(O_X)$ a differential operator with $n > 0$. Then for any section $a \in \Gamma(X, O_X)$, the function $D_a \in \mathrm{End}_{f^{-1}(O_S)(O_X)}$ defined by*

$$D_a(t) := D(at) - aD(t) \qquad \forall\ t \in \Gamma(X, O_X)$$

*is a differential operator of order $\leq n-1$.*

Lemma 3.0.8. *Let $f : X \to S$ be a morphism of schemes, $U \subseteq X$ an open subset, $D$ be a section of the sheaf $\mathcal{D}iff^n_{X/S}(O_X)$ on $U$, $(E,\nabla)$ be a $D$-module, $t \in \Gamma(U, E)$, $a \in \Gamma(U, O_X)$, then we have the following Leibniz's rule:*

$$\nabla(D)(at) = a\nabla(D) + \nabla(D_a)(t).$$

Remark 3.0.9. The proof of the above lemma is immediate from the definition. But it is quite handy if one wants to prove something using induction on the order of the differential operators. For example, one can show that if $(E,\nabla)$ is a differential operator over $X/S$, then $\nabla$ factors though the natural inclusion $\mathcal{D}iff_{X/S}(E) \subseteq \mathcal{E}nd_{f^{-1}O_S}(E)$, where $\mathcal{D}iff_{X/S}(E)$ is defined similarly as $\mathcal{D}iff_{X/S}(O_X)$: one takes the tensor product $\mathcal{P}^n_{X/S} \otimes_{O_X} E$ via the second projection $O_X \to \mathcal{P}^n_{X/S}$ and views the tensor product as an $O_X$-module via the first projection, then one defines $\mathcal{D}iff^n_{X/S}(E) := \mathcal{H}om_{O_X}(\mathcal{P}^n_{X/S} \otimes_{O_X} E, E)$, now one passes to the limit and defines $\mathcal{D}iff_{X/S}(E)$.

Proposition 3.0.10. *Let $X$ be a scheme locally of finite type over a field $k$, $E$ be a coherent sheaf on $X$. If $E$ carries a structure of a $D$-module over $X/k$ then $E$ is necessarily a vector bundle.*

Definition 3.0.11. Let $f : X \to S$ be a morphism of schemes, $\mathcal{D}er_{X/S}(O_X)$ be the sheaf of derivations of $O_X$, $E$ a sheaf of $O_X$-module on $X$. An integrable connection $\nabla$ on $E$ is an $O_X$-linear map

$$\nabla : \mathcal{D}er_{X/S}(O_X) \to \mathcal{E}nd_{f^{-1}O_S}(E)$$

satisfying the following conditions:

(1) $\nabla(D)(fs) = f\nabla(D)(s) + D(f)s \qquad (\forall f \in O_X, D \in \mathcal{D}er_{X/S}(O_X), s \in E)$
(2) $\nabla([D_1, D_2]) = [\nabla(D_1), \nabla(D_2)] \qquad (\forall D_1, D_2 \in \mathcal{D}er_{X/S}(O_X), s \in E).$

Remark 3.0.12. There is a tensor structure in the category of integrable connections over $X/S$. Given $(E, \nabla_E)$ and $(F, \nabla_F)$ we can define an integrable connection

$$\nabla_{E\otimes_{O_X} F} : \mathcal{D}er_{X/S}(O_X) \to \mathcal{E}nd_{f^{-1}O_S}(E \otimes_{O_X} F)$$

as follows: for any $e \in E$, $f \in F$ and $D \in \mathcal{D}er_{X/S}(O_X)$,

$$e \otimes f \mapsto \nabla_E(D)(e) \otimes f + e \otimes \nabla_F(D)(f).$$

We can also define for each two integrable connections $(E, \nabla_E)$ and $(F, \nabla_F)$ an integrable connection

$$\nabla_{\mathcal{H}om_{O_X}(E,F)} : \mathcal{D}er_{X/S}(O_X) \to \mathcal{E}nd_{f^{-1}O_S}(\mathcal{H}om_{O_X}(E, F))$$

as follows: for any $D \in \mathcal{D}er_{X/S}(O_X)$, $\phi \in \mathcal{H}om_{O_X}(E, F)$ and $e \in E$, we have

$$(\nabla_{\mathcal{H}om_{O_X}(E,F)}(D)(\phi))(e) = \nabla_F(D)(\phi(e)) - \phi(\nabla_E(D)(e)).$$

Note that if $X$ is locally noetherian then the above definition gives an internal Hom functor in the category of coherent sheaves with integrable connections.

The neutral object in this category is of the form $(O_X, \nabla)$ where $\nabla$ is the natural inclusion

$$\nabla : \mathcal{D}er_{X/S}(O_X) \to \mathcal{E}nd_{f^{-1}O_S}(O_X)$$

sending a derivation to itself (viewing as an $f^{-1}O_S$-linear endomorphism of $O_X$).

Furthermore, one can also define the wedge product of an integrable connections $(E, \nabla_E)$ with ifself

$$\nabla_{E\wedge E} : \mathcal{D}er_{X/S}(O_X) \to \mathcal{E}nd_{f^{-1}O_S}(E \wedge E)$$

as follows: for any $e \in E$, $f \in E$ and $D \in \mathcal{D}er_{X/S}(O_X)$,

$$e \wedge f \mapsto \nabla_E(D)(e) \wedge f + e \wedge \nabla_E(D)(f).$$

This is well defined as one can check that for any $e \in E$,

$$\begin{aligned} a(e \otimes e) &\mapsto \nabla_E(D)(ae) \wedge e + ae \wedge \nabla_E(D)(e) \\ &= a(\nabla_E(D)(e) \wedge e + e \wedge \nabla_E(D)(e)) + D(a)e \wedge e \\ &= 0. \end{aligned}$$

Proposition 3.0.13. *If $f : X \to S$ is smooth, then an integrable connection is equivalent to an $f^{-1}O_S$-linear map $\nabla : E \to E \otimes_{O_X} \Omega^1_{X/S}$ satisfying:*

*(1) $\nabla(fs) = f\nabla(s) + s \otimes df \ \forall f \in O_X, s \in E$.*
*(2) If $\nabla_1$ denotes the map*

$$\nabla_1 : E \otimes_{O_X} \Omega^1_{X/S} \to E \otimes_{O_X} \Omega^2_{X/S}$$

*sending $s \otimes \omega \mapsto s \otimes d\omega - \nabla(s) \wedge \omega$ for all $s \in E, \omega \in \Omega^1_{X/S}$, then $\nabla \circ \nabla_1 = 0$.*

Proposition 3.0.14. *If $f : X \to S$ is smooth, and $S$ is a scheme over $\mathbb{Q}$, then the category of integrable connections over $X/S$ is equivalent to the category of $D$-modules over $X/S$.*

Remark. The idea of the proof of the above proposition is quite simple. This is just because in characteristic 0 the differential operators of order $\leq 1$ generate the whole ring of differential operators as a non-commutative left $O_X$-algebra.

Theorem 3.0.15. *Let $X$ be a smooth scheme over a field $k$ of characteristic 0, then $\mathsf{Mod}_c(D_{X/k})$ is a rigid $k$-linear abelian tensor category. If $X$ is connected and $x \in X(k)$ then $\mathsf{Mod}_c(D_{X/k})$ equipped with the functor $E \mapsto E|_x$ from $\mathsf{Mod}_c(D_{X/k})$ to $\mathrm{Vec}_k$ is a neutral Tannakian category whose Tannakian group $\pi^{alg}(X, x)$ is called the algebraic fundamental group.*

Proof. This follows immediately from Remark 1.54, Proposition 1.52 and Proposition 1.56. ☐

Definition 3.0.16. Let $f : X \to S$ be a morphism of schemes, $S$ is a scheme over $\mathbb{F}_p$. For $i \in \mathbb{N}$, we have the relative Frobenius $\phi_i : X^{(i)} \to X^{(i+1)}$ starting with $X^{(0)} = X$. A stratified bundle $(E^{(i)}, \sigma^{(i)}, i \in \mathbb{N})$ consists of a sequence of coherent sheaves $E^{(i)}$ on $X^{(i)}$, and a sequence of $O_{X^{(i)}}$-isomorphisms

$$\sigma_i : E^{(i)} \to \phi_i^* E^{(i+1)}$$

for all $i \in \mathbb{N}$. Now we obtained a category $\mathrm{Strat}(X/S)$ whose objects consist of stratified bundles, whose morphisms between two objects

$$\mathrm{Hom}((E^{(i)}, \sigma^{(i)}, i \in \mathbb{N}), (F^{(i)}, \tau^{(i)}, i \in \mathbb{N}))$$

is the set of morphisms $E^{(i)} \to F^{(i)}$ for $i \in \mathbb{N}$ which are compatible with $\sigma^{(i)}$ and $\tau^{(i)}$ in a natural way. We denote by $\mathrm{Strat}_v(X/S)$ the full subcategory of $\mathrm{Strat}(X/S)$ consisting of objects $(E^{(i)}, \sigma^{(i)}, i \in \mathbb{N})$ where $E^{(i)}$ are vector bundles.

Remark 3.0.17. There is an obvious tensor structure on $\mathrm{Strat}(X/S)$. We can also define the Hom object for two objects in $\mathrm{Strat}(X/S)$. If $X$ is locally noetherian then this defines the internal Hom functor in $\mathrm{Strat}(X/S)$. The identity object is of the form $(E^{(i)}, \sigma^{(i)}, i \in \mathbb{N})$ where $E^{(i)} = O_X$ for all $i \in \mathbb{N}$ and $\sigma^{(i)}$ are all identities.

Theorem 3.0.18. [**Gies**][Theorem 1.3] *Let $f : X \to S$ be a smooth morphism of schemes, $S$ is a scheme over $\mathbb{F}_p$. Then there is an equivalence of categories between $\mathrm{Strat}_v(X/S)$ and the full subcategory $\mathsf{Mod}_v(D_{X/S})$ of $\mathsf{Mod}_c(D_{X/S})$ consisting of $D$-modules of the form $(E, \nabla)$ where $E$ is a vector bundle.*

Proof. We will set up two functors:

$$F : \mathsf{Mod}_c(D_{X/S}) \to \mathrm{Strat}_v(X/S) \qquad \text{and} \qquad G : \mathrm{Strat}_v(X/S) \to \mathsf{Mod}_c(D_{X/S})$$

which are quasi-inverse to each other.

The construction of $F$ is proceeded by induction. Suppose we have an object $(E, \nabla) \in \mathsf{Mod}_v(D_{X/S})$. We set $E^{(i)}$ to be the subsheaf of abelian groups of $E$ consisting of sections which are annihilated by all $\nabla(D)$ where $D$ is a differential operator order $< p^i$ and $D(1) = 0$. Since the action of differential operators of order $< p^i$ on $E$ commutates with the action of $O_{X^{(i)}}$ via the relative Frobenius, $E^{(i)}$ carries a structure of an $O_{X^{(i)}}$-module. Suppose $E^{(i)}$ is a vector bundle, we want to show that $E^{(i+1)}$ is also a vector bundle and there is an isomorphism $\sigma_i : E^{(i)} \to \phi_i^* E^{(i+1)}$. We first define a connection $\nabla^{(i)}$ on $E^{(i)}$ over $X^{(i)}/S$ and then use Cartier descent. $\nabla^{(i)}$ can be defined as follows: Let $U$ be an open affine of $U$ which admits local coordinates $(x_1, x_2, \cdots, x_n)$. and let $D = \sum f_k(\partial/\partial x_k)$ be a derivation over $U/S$ and $D'$ be a differential operator of degree $\leq p^i$ so that

$$D'(f^{p^i}) = (D(f))^{p^i}.$$

Such an operator could be given by

$$\sum_k f_k^p \frac{\partial^{p^i}/\partial x_k^{p^i}}{p^i!}.$$

If $D''$ is another operator of degree $\leq p^i$ satisfying $D'(f^{p^i}) = (D(f))^{p^i}$, then $D' - D''$ is in the ring generated by operators of degree $< p^i$. Thus if we set $\nabla^{(i)}(D)(s) := \nabla(D')(s)$ then this $\nabla^{(i)}$ is well defined. Since $\sum_k f_k^p \frac{\partial^{p^i}/\partial x_k^{p^i}}{p^i!}$ commutes with all monomials $D_I$ with $|I| < p^i$, so $\nabla(D)(s)$ is a section of $E^{(i)}$, and hence $\nabla^{(i)}(D) \in \mathcal{E}nd_{f^{-1}O_S}(E^{(i)})$. Now let $f^{(i)} \in O_{X^{(i)}}$ then we write $f$ for the image of $f^{(i)}$ under the relative Frobenius $O_{X^{(i)}} \to O_X$. Then we have:

$$\begin{aligned}
\nabla^{(i)}(D)(f^{(i)}s) &= \nabla(D')(fs) \\
&= f\nabla(D')(s) + \nabla(D'_f)(s) \\
&= f^{(i)}\nabla(D')(s) + (D(f^{(i)}))s,
\end{aligned}$$

where the last equation comes from

$$\begin{aligned}
D'_f(g) &= D'(fg) - fD'(g) \\
&= D'_g(f) + gD'(f) - fD'(g) \\
&= fD'_g(1) + gD'(f) - fD'(g) \\
&= fgD'(1) - fD'(g) + gD'(f) - fD'(g) \\
&= gD(f^{(i)}).
\end{aligned}$$

It is not hard to check that $\nabla^{(i)}(D^p) = (\nabla^{(i)}(D))^p$, and that the sheaf of all sections annihilated by $\nabla^{(i)}$ is precisely $E^{(i+1)}$. Now one can apply Cartier descent and get the functor $F$.

Conversely, if we have an object $(E^{(i)}, \sigma^{(i)}, i \in \mathbb{N}) \in \mathrm{Strat}_v(X/S)$. If $U \subseteq X$ is an open affine where $s_1, \cdots, s_r \in \Gamma(U, E^{(i)})$ give a trivialization of $E^{(i)}$. Then any section $s \in E$ can be written as

$$s = \sum_k f_k \varphi_i(s_k)$$

where $\varphi_i$ is the canonical imbedding $E^{(i)} \to E^{(0)}$. If $D$ is a differential operator of order $< p^i$, then we set

$$\nabla(D)(s) = \sum_k D(f_k)\varphi_i(s_k).$$

Since the operation of $D$ on $O_X$ commutates with the action of $O_{X^{(i)}}$ via the relative Frobenius $O_{X^{(i)}} \to O_X$, the construction above does not depend on the choice of the basis. One can check that the above construction really gives us a $D$-module structure on $E^{(0)}$. This defines $G$. Now it is not hard to check that $F$ and $G$ are quasi-inverse to each other. $\square$

PROPOSITION 3.0.19. *Let $X$ be a smooth scheme over a field $k$ of characteristic $p > 0$. Then* $\mathrm{Strat}_v(X/k) = \mathrm{Strat}(X/k)$.

THEOREM 3.0.20. *Let $X$ be a smooth connected scheme over a field $k$ of characteristic $p > 0$, $x \in X(k)$. Then* $\mathrm{Strat}(X/k)$ *is a $k$-linear rigid abelian tensor category and the functor*

$$(E^{(i)}, \sigma^{(i)}, i \in \mathbb{N}) \mapsto E^{(0)}|_x$$

*is a fibre functor. So* $\mathrm{Strat}(X/k)$ *equipped with this fibre functor is a neutral Tannakian category whose Tannakian group is denoted by $\pi^{alg}(X, x)$. This group scheme is called the algebraic fundamental group.*

CHAPTER 2

# The Homotopy Sequence for Nori's Fundamental Group

In this chapter we will study the Homotopy sequence for Nori's fundamental group. The question is the following: If we have a separable proper morphism $X \to S$ with geometrically connected fibres between two reduced connected locally noetherian schemes over a field $k$, $x \in X(k)$, $s \in S(k)$, $f(x) = s$, then by functoriality of Nori's fundamental group we get a sequence of maps in the category of $k$-group schemes:

$$\pi^N(X_s, x) \to \pi^N(X, x) \to \pi^N(S, s) \to 1.$$

We have already known from Chapter 1 §2 that the above sequence is exact on the right, so the question is whether or not it is exact in the middle. We are not trying to show the arrow at the very left is injective because that is already false for étale fundamental group and the topological fundamental group, that is a place for the higher homotopy groups.

## 1. The general criterion

In this section, we will give a necessary sufficient condition for the exactness of the homotopy sequence for the general base (i.e. we will not assume the base scheme is proper or smooth). And then we will apply the necessary sufficient condition to the étale quotient of Nori's fundamental group and get the exactness of the homotopy sequence there. This is can thought as a different proof of the exactness of the étale fundamental group.

Definition 1.0.21. Let $X$ be a reduced connected scheme over a field $k$, $x \in X(k)$ be a rational point. We call a triple $(P, G, p) \in N(X, x)$ a $G$-saturated torsor if the canonical map $\pi^N(X, x) \to G$ is surjective.

Remark. Here we are using the terminology in [**EHV**][Definition 3.2], where they defined a $G$-saturated bundle to be a pointed torsor $(P, G, p) \in N(X, x)$ with the property that $O_P(P) = k$. Nori has proved in [**Nori**][Part I, Chapter II, Proposition 3] that if $X$ is (in addition) proper then the two definitions above are equivalent, where he called a $G$-saturated torsor "reduced" [**Nori**][Part I, Chapter II, Definition 3].

Definition 1.0.22. Let $f : X \to S$ be a map of schemes, $\mathcal{F}$ be a sheaf of $O_X$-modules, $s : \mathrm{Spec}\,(\kappa(s)) \hookrightarrow S$ a point, then we get a Cartesian diagram:

$$\begin{tikzcd} X_s \arrow[r, "t"] \arrow[d, "g"] & X \arrow[d, "f"] \\ \mathrm{Spec}\,(\kappa(s)) \arrow[r, "s"] & S \end{tikzcd} .$$

We say $\mathcal{F}$ satisfies base change at $s$ if the canonical map

$$s^*f_*\mathcal{F} \to g_*t^*\mathcal{F}$$

is surjective. Note that if $f$ is proper, $S$ is locally noetherian, $\mathcal{F}$ is coherent and flat over $S$ then $\mathcal{F}$ satisfies base change at $s$ if and only if the above canonical map is an isomorphism (see [**Hart**][Chapter III, Theorem 12.11]).

THEOREM 1.0.23. (H.Esnault, P.H.Hai , E.Viehweg) *Let $f : X \to S$ be a separable proper morphism with geometrically connected fibres between two reduced connected locally noetherian schemes over a perfect field $k$. We suppose further that $S$ is irreducible. Let $x \in X(k)$, $s \in S(k)$ and assume $f(x) = s$. Then the following conditions are equivalent:*

*(1) the sequence*

$$\pi^N(X_s, x) \to \pi^N(X, x) \to \pi^N(S, s) \to 1$$

*is exact;*

*(2) for any $G$-saturated torsor $(P, G, p)$ with structure map $\pi : P \to X$, $\pi_*O_P$ satisfies base change at $s$ and the image of the composition $\pi^N(X_s, x) \to \pi^N(X, x) \to G$ is a normal subgroup of $G$;*

*(3) for any $G$-saturated torsor $(P, G, p)$ with structure map $\pi : P \to X$, $\pi_*O_P$ satisfies base change at $s$ and there is a $G'$-saturated torsor $\pi' : P' \to S$ together with a morphism $(P, G) \xrightarrow{\theta} (P', G')$ satisfy that the $\theta$-induced map $(\pi'_*O_{P'})_s \to (f_*\pi_*O_P)_s$ is an isomorphism.*

PROOF. "(1) $\implies$ (2)" If the homotopy sequence is exact then clearly the image of $\pi^N(X_s, x) \to \pi^N(X, x) \to G$ (which is denoted by $H$) is normal in $G$. The exactness also gives us a commutative diagram

$$\begin{array}{ccc} \pi^N(X, x) & \longrightarrow & \pi^N(S, s) \\ \downarrow & & \downarrow \\ G & \longrightarrow & G/H \end{array} .$$

This commutative diagram gives us a $G/H$-saturated torsor $(P', G/H, p')$ over $S$ and a morphism in $N(X, x)$:

$$\lambda : (P, G, p) \to (P' \times_S X, G/H, p' \times_S X) \cong (P/H, G/H, p).$$

Let $W'$ be the push forward of the structure sheaf of $P'$ to $S$, $V := \pi_*O_P$, $W := f^*W'$. Let $\lambda^* : W \to V$ be the map induced by $\lambda$. If we pull-back $\lambda^*$ to $X_s$ then we get a morphism in the category of essentially finite vector bundles because $V|_{X_s}$ (resp.$W|_{X_s}$) is the 0-th direct image of the structure sheaf of the torsor $P \times_X X_s$ (resp.$P' \times_S X_s$). From [**Nori**][Part I, Chapter I, Proposition 2.9], this $\lambda^*$ corresponds, via Tannakian duality, to the morphism

$$k[G]^{\pi^N(X_s,x)} = k[G]^H = k[G/H] \to k[G]$$

in the category of $\mathsf{Rep}_k(\pi^N(X_s, x))$. Hence $W|_{X_s}$ is the maximal trivial subbundle of $V|_{X_s}$. But $H^0(X_s, V|_{X_s}) \otimes_k O_{X_s} \subseteq V|_{X_s}$ is the maximal trivial sub embedding (see lemma 2.3

below), thus the canonical map

$$W|_{X_s} = H^0(X_s, W|_{X_s}) \otimes_k O_{X_s} \to H^0(X_s, V|_{X_s}) \otimes_k O_{X_s}$$

is an isomorphism. But note that the above map factors $W|_{X_s} \to f^*f_*V|_{X_s}$. This implies $f^*f_*V|_{X_s} \to H^0(X_s, V|_{X_s}) \otimes_k O_{X_s}$ is an isomorphism, so base change is satisfied.

"(2) $\Longrightarrow$ (3)" Let $H \subseteq G$ be the image of the composition $\pi^N(X_s, x) \to \pi^N(X, x) \to G$. Since it is normal we get a $G/H$-torsor $P/H$ on $X$. If $W$ is the push-forward of the structure sheaf of $P/H$ to $X$ and $V := \pi_* O_P$, then we know from our assumption that $W$ and $V$ satisfy base change at $s$. Let $\lambda : W \hookrightarrow V$ be the imbedding induced $P \to P/H$, then we have the following commutative diagram of sheaves on $X_s$:

$$\begin{tikzcd}
f^*f_*W|_{X_s} \arrow[r, "a_1"] \arrow[d, "f^*f_*\lambda"] & H^0(X_s, W|_{X_s}) \otimes_k O_{X_s} \arrow[r, "a_2"] \arrow[d, "H^0(X_s, \lambda|_{X_s})"] & W|_{X_s} \arrow[d, "\lambda|_{X_s}"] \\
f^*f_*V|_{X_s} \arrow[r, "a_3"] & H^0(X_s, V|_{X_s}) \otimes_k O_{X_s} \arrow[r, "a_4"] & V|_{X_s}
\end{tikzcd} .$$

By base change $a_1, a_3$ are isomorphisms. Since $\lambda|_{X_s}$ corresponds via Tannakian duality to $k[G]^H \hookrightarrow k[G]$ (in the category $\mathsf{Rep}_k(\pi^N(X_s, x))$), $W|_{X_s}$ is imbedded as the maximal trivial subbundle of $V|_{X_s}$. Hence $a_2$ and $H^0(X_s, \lambda|_{X_s})$ are isomorphisms. So $f^*f_*\lambda$ is also an isomorphism. In particular

$$(f_*\lambda)_x : (f_*W)_x \to (f_*V)_x$$

is an isomorphism. Let $r \in \mathbb{N}$ be the rank of $W$. For any point $t \in S$, since

$$H^0(X_t, W|_{X_t}) \otimes_k O_{X_t} \to W|_{X_t}$$

is always an imbedding (lemma 2.3), we have $\dim_k(H^0(X_t, W|_{X_t})) \leq r$. But on the other hand, since $W$ satisfies base change at $s$, $r = \dim_k(H^0(X_s, W|_{X_s}))$ reaches the minimal dimension (the dimension at the generic point), so by semi-continuity theorem we have

$$\dim_k(H^0(X_t, W|_{X_t})) \geq \dim_k(H^0(X_s, W|_{X_s})) = r.$$

This implies $H^0(X_t, W|_{X_t})$ has constant dimension $r$, and hence $W$ satisfies base change all over $S$. So $f_*W$ a vector bundle. Since $f^*f_*W \to W$ is injective after restricting to all the points of $X$, we have it is an embedding as a subbundle (i.e. injective and locally split). But since $a_1, a_2$ are isomorphisms, we have $f^*f_*W \to W$ is an isomorphism. Now we can check easily that $\mathrm{Spec}\,(f_*W) \to S$ with the canonical $G/H$-action induced from $P/H$ is an FPQC-torsor which satisfies all our conditions in (3).

"(3) $\Longrightarrow$ (1)" Let $N$ be the image of $\mathrm{Ker}(\pi^N(f))$ in $G$ (where $\pi^N(f)$ is the map $\pi^N(X, x) \to \pi^N(S, s)$), $N'$ be the kernel of $G \to G'$, and $H \subseteq G$ be the image of the composition $\pi^N(X_s, x) \to \pi^N(X, x) \to G$. We also write $W := \pi'_* O_{P'}$ and $V := \pi_* O_P$. We first note that the $\theta$-induced map $f^*W|_{X_s} \to V|_{X_s}$ corresponds to $k[G/N'] \to k[G]$ in $\mathsf{Rep}_k(\pi^N(X_s, x))$. But from base change of $V$ and the fact that the $\theta$-induced map $W_x \to (f_*V)_x$ is an isomorphism we know that $f^*W|_{X_s} \to V|_{X_s}$ should be the same as $H^0(X_s, V|_{X_s}) \otimes_k O_{X_s} \to V|_{X_s}$ as subobjects. Thus the canonical imbedding $k[G/N'] \hookrightarrow k[G/H]$ should be an isomorphism. Hence $N' = H$ as subgroups. But since we have

$H \subseteq N \subseteq N'$, so $H = N$ as well. Because the equality holds for all $G$-saturated torsor $(P, G, p)$, we have $\pi^N(X_s, x) \to \mathrm{Ker}(\pi^N(f))$ is surjective. This completes the proof. $\square$

LEMMA 1.0.24. *If $X$ is a reduced connected proper scheme over a perfect field $k$ with a rational point $x \in X(k)$, then for any essentially finite vector bundle $V$ on $X$ the canonical morphism $\Gamma(X, V) \otimes_k O_X \to V$ imbeds $\Gamma(X, V) \otimes_k O_X$ as the maximal trivial subbundle of $V$.*

PROOF. Let $\mathrm{Ess}(X)$ be the category of essentially finite vector bunldes, $\omega_x : \mathrm{Ess}(X) \to \mathrm{Vec}_k$ be the fibre functor. Then applying $\omega_x$ to the canonical morphism $\Gamma(X, V) \otimes_k O_X \to V$ we get $\mathrm{Hom}_{O_X}(O_X, V) \cong \Gamma(X, V) \to V_x \otimes_{O_{X,x}} k = \omega_x(V)$. But note that we have $\mathrm{Hom}_{O_X}(O_X, V) \cong \mathrm{Hom}_{\pi^N(X,x)}(k, \omega_x(V))$ where $k$ stands for the dim 1 vector space with trivial $\pi^N(X, x)$ action. One checks readily that under these isomorphisms we get exactly the canonical injection $\mathrm{Hom}_{\pi^N(X,x)}(k, \omega_x(V)) \to \omega_x(V)$ sending any morphism $k \to \omega_x(V)$ to the image of $1 \in k$. Since this map imbeds $\mathrm{Hom}_{\pi^N(X,x)}(k, \omega_x(V))$ as the maximal trivial sub of $\omega_x(V)$. Using Tannakian duality we get our result. $\square$

### 1.1. Application to the étale quotient.

DEFINITION 1.1.1. Let $X$ be a connected reduced locally noetherian scheme over a perfect field $k$ which admits a rational point $x \in X(k)$. Let $N^{\text{ét}}(X, x)$ be the full subcategory of $N(X, x)$ whose objects consist of those $(P, G, p)$ with $G$ finite étale. We define the étale quotient of $\pi^N(X, x)$ to be $\pi^{\text{ét}}(X, x) := \varprojlim_{N^{\text{ét}}(X,x)} G$. We have an obvious surjection: $\pi^N(X, x) \twoheadrightarrow \pi^{\text{ét}}(X, x)$.

LEMMA 1.1.2. *Let $X$ be a connected reduced scheme over a perfect field $k$ which admits a rational point $x \in X(k)$. Let $(P, G, p)$ be an étale torsor over $(X, x)$. This torsor is $G$-saturated if and only if $P$ is connected.*

PROOF. Since $P$ has a rational point so connectedness is equivalent to geometrical connectedness, and also the formation of Nori's fundamental group is compatible with separable field extensions, thus we can reduce to the case when $k$ is algebraically closed.

"$\Longrightarrow$" Let's take $Q \subseteq P$ to be the connected component of $P$ containing $p$. Now $G$ is an abstract group we can write the action $\rho : P \times_k G \to P$ as $\coprod_G P \to P$ where each component in the direct union is mapped to $P$ via a unique element in $G$. Since $P$ is an $G$-torsor we have the following cartesian diagram:

$$\begin{tikzcd} \coprod_G P \arrow[r, "\rho"] \arrow[d, "id^G"] & P \arrow[d] \\ P \arrow[r] & X \end{tikzcd} .$$

If we let $H \subseteq G$ be the maximal subgroup of $G$ which fix $Q$, then we can see by definition that $Q \times_k H \subseteq P \times_k G$ is the intersection of $\rho^{-1}(Q)$ and $(id^G)^{-1}(Q)$. Thus the square

$$\begin{tikzcd} \coprod_H Q \arrow[r, "\rho"] \arrow[d, "id^H"] & Q \arrow[d] \\ Q \arrow[r] & X \end{tikzcd}$$

is cartesian. Hence $Q$ is an $H$-torsor. But from the assumption the imbedding $H \to G$ should be surjective. This tells us $H = G$. But then the map of $G$-torsors $Q \subseteq P$ should also be an isomorphism. So $P$ is connected.

"$\Longleftarrow$" Let $(P', G', p') \to (P, G, p)$ be any morphism in $N(X, x)$. Since $P \to X$ is étale, we know $P' \to P$ is finite flat. Thus the image must be both open and closed, and hence it must be the whole of $P$. But if we pull-back the surjective map $P' \to P$ via $x \in X(k)$, we will get the group homomorphism $G' \to G$. Thus this homorphism must be surjective. Since $(P', G', p')$ is taken arbitrarily, it actually shows that $(P, G, p)$ is $G$-saturated. $\square$

THEOREM 1.1.3. *Let $f : X \to S$ be a separable proper morphism with geometrically connected fibres between two reduced connected locally noetherian schemes over a perfect field $k$. Let $x \in X(k)$, $s \in S(k)$ and assume $f(x) = s$. Then the homotopy sequence:*

$$\pi^{\acute{e}t}(X_s, x) \to \pi^{\acute{e}t}(X, x) \to \pi^{\acute{e}t}(S, s) \to 1$$

*is exact.*

PROOF. Without loss of generality one may assume $k = \bar{k}$ [**Nori**][Part I, Chapter II, Proposition 5]. Now let $(P, G, p)$ be a $G$-saturated étale torsor over $X$, $\pi : P \to X$ be the structure map $V := \pi_* O_P$. Let $P \xrightarrow{\phi} Q \xrightarrow{\varpi} S$ be the stein factorization of the proper map $P \xrightarrow{\pi} X \xrightarrow{f} S$. Since $f \circ \pi$ is proper separable $\varpi$ is finite étale. Thus $\phi$ is proper separable surjective with geometrically connected fibres. But then the pull back $\phi_s : P_s \to Q_s$ along the rational point $s \hookrightarrow S$ is also proper separable surjective with geometrically connected fibres. Hence $O_{Q_s} \to (\phi_s)_* O_{P_s}$ is an isomorphism. This tells us base change is satisfied for $P$ at $s$.

The action $P \times_k G \to P$ induces a map $V \to V \otimes_k k[G]$. Push it to $S$ we get $f_* V \to f_* V \otimes_k k[G]$. Thus there is an action of $G$ on $Q$ which makes $\phi$ $G$-equivariant. If we pull back the map $P \to Q \times_S X$ along the rational point $x \in X(k)$, we get a $G$-equivariant map $t : G \to G'$ (where the identity point $e$ of $G$ comes from $p$ and $G'$ is a $G$-set with a distinguished point $q$). One checks readily that $H := t^{-1}(q)$ is the stabilizer of $t(e)$, hence a subgroup of $G$. Now let $h \in H$ be an element. Considering the $S$-isomorphism $Q \to Q$ induced by $h$. Evidently $h$ sends $q$ to $q$, and since $Q$ is a connected finite étale cover of $S$, the $S$-isomorphism induced by $h$ must be the identity. Hence $H$ acts trivially on $Q$ and in particular it also acts trivially on $G'$. So for any $x \in G$, we have $t(e)xhx^{-1} = t(x)hx^{-1} = t(x)x^{-1} = t(xx^{-1}) = t(e)$. As a consequence, $H$ is a normal subgroup of $G$. But since $t : G \to G'$ is faithfully flat, we actually know that $G'$ is the quotient of $G$ by $H$ (and $t$ is the quotient map). Thus we get a commutative diagram:

$$\begin{tikzcd} P \times_k G \arrow[r, "\cong"] \arrow[d] & P \times_X P . \arrow[d] \\ Q \times_k G' \arrow[r, "\rho"] & Q \times_S Q \end{tikzcd}$$

Let $r$ be the degree of the connected finite étale cover $\varpi : Q \to S$. Then one sees easily that both $Q \times_k G'$ and $Q \times_S Q$ are finite étale of degree $r$. This shows that the $Q$-morphism $\rho$ is finite étale of degree 1, and hence an isomorphism. Now $\varpi : Q \to S$ has a structure of a $G'$-torsor which satisfies all the conditions in (3) of our main theorem. So we can use the same argument we have used in "(3) $\Longrightarrow$ (1)" to conclude our theorem. ☐

## 2. The proper case

Under the properness assumption for the base $S$, the necessary sufficient condition for the exactness of the homotopy sequence is a kind of neat:

THEOREM 2.0.4. (H.Esnault, P.H.Hai , E.Viehweg) *Let $f : X \to S$ be a proper separable morphism with geometrically connected fibres between two reduced connected proper schemes over a perfect field $k$, $x \in X(k)$, $s \in S(k)$, $f(x) = s$. Assume further that $S$ is irreducible. Then the homotopy sequence*

$$\pi^N(X_s, x) \to \pi^N(X, x) \to \pi^N(S, s) \to 1$$

*is exact if and only if for any $G$-saturated torsor $(P, G, p) \in N(X, x)$ with structure map $\pi : P \to X$, $V := \pi_* O_P$ satisfies base change at $s$ and $f_* V$ is essentially finite.*

PROOF. " $\Longleftarrow$ " Since $f_*V$ satisfies base change, the canonical map $f^*f_*V \to V$ is of the form

$$\Gamma(X_s, V|_{X_s}) \otimes_k O_{X_s} \to V|_{X_s}$$

after restricting to the fibre $X_s$. Because $f^*f_*V \to V$ is a map of essentially finite vector bundles, the kernel of it is also a vector bundle. But the kernel is trivial on $X_s$, so the kernel itself is trivial. Thus $f^*f_*V \subseteq V$ is a subobject in the category of essentially finite vector bundles on $X$ and it becomes the maximal trivial suboject after restricting to $X_s$. Now let $G'$ be the Tannakian group of the sub Tannakian category of $\mathrm{Ess}(S)$ generated by $f_*V$. The imbedding $f^*f_*V \to V$ gives us a surjection $\lambda : G \to G'$. Let $H$ be the kernel of $\lambda$. Then $f^*f_*V \to V$ corresponds via Tannakian duality to an inclusion $M \subseteq k[G]$ in $\mathsf{Rep}_k(G)$. Note that since $M$ comes from an object in $\mathsf{Rep}_k(G')$ via $\lambda : G \to G'$, so $M \subseteq k[G]$ factors through the inclusion $k[G]^H \subseteq k[G]$. On the other hand, since we have a surjection $\pi^N(S, s) \to G'$, by [**Nori**][Chapter I, Proposition 3.11] we have a $G'$-saturated torsor $(P', G', p') \in N(S, s)$ with a map

$$\theta : (P, G, p) \to f^*(P', G', p')$$

in $N(X, x)$ extending $\lambda$. Let $V' := \pi'_* O_{P'}$, $\pi' : P' \to S$. Then since $P \to P' \cong P/H$ is faithfully flat, $f^*V' \subseteq V$ is a subbundle, and this subbundle corresponds via Tannakian duality to the inclusion $k[G]^H \subseteq k[G]$. But clearly $f^*V' \subseteq V$ factors $f^*f_*V \to V$, so

$k[G]^H \subseteq k[G]$ factors $M \subseteq k[G]$, which means $k[G]^H = M$. So we have $V' \cong f_*V$. Now the triple $(P', G', p')$ satisfies all our conditions in Theorem 1.0.23 (3), so we get the exact sequence.

" $\Longrightarrow$ " By Theorem 1.0.23 we have a $G'$-saturated torsor $(P', G', p') \in N(S, s)$ and a morphism

$$\theta : (P, G, p) \to f^*(P', G', p') \in N(X, x)$$

such that the induced map $V'_s \to (f_*V)_s$ is an isomorphism, where $V' := \pi'_* O_{P'}$ and $\pi' : P' \to S$ is the structure map. Because $V$ satisfies base change at $s$, there is a neighborhood $s \in U$ such that $f_*V$ is a vector bundle on $U$ and the adjunction map $f^*f_*V \to V$ is a sub bundle (locally split) on $f^{-1}(U)$. But $f^*V' \to V$ is a sub vector bundle and $f^*V' \to V$ factors through the adjunction map, so $f^*V' \to f^*f_*V$ is a sub vector bundle on $f^{-1}(U)$. Since $V'_s \cong (f_*V)_s$, $f^*V' \to f^*f_*V$ is an isomorphism on $f^{-1}(U)$. Hence the injective map $V' \to f_*V$ is also an isomorphism on $U$. Now by [**?**][Théorème 7.7.6] there is a coherent sheaf $\mathcal{Q}$ on $S$ such that

$$f_*(V/f^*V') \cong \mathcal{H}om_{O_S}(\mathcal{Q}, O_S).$$

Since locally $\mathcal{H}om_{O_S}(\mathcal{Q}, O_S)$ is contained in a vector bundle and we have

$$f_*V/V' \subseteq f_*(V/f^*V') = \mathcal{H}om_{O_S}(\mathcal{Q}, O_S),$$

so if there is $t \in S \setminus U$ such that $(f_*V/V')_t \neq 0$, then we can choose an open affine $t \in \mathrm{Spec}\,(A) \subseteq S$ such that

$$(f_*V/V')|_{\mathrm{Spec}\,(A)} \subseteq \bigoplus_{i=0}^{n} A_i,$$

where $A_i$ is a rank 1 free $A$-module for all $0 \leq i \leq n$. Notice that since $S$ is integral $\mathrm{Spec}\,(A)$ is non-empty, so $A$ is an integral ring. This implies $f_*V/V'$ is non-zero at the generic point which contradicts to the fact that $f_*V/V'$ has support in $S \setminus U$. So $V' \to f_*V$ is an isomorphism on $S$. But $V'$ is certainly essentially finite. This completes the proof. $\square$

### 2.1. Application to the Künneth formula.

DEFINITION 2.1.1. Let $X$ be a reduced connected scheme over a field $k$ with a rational point $x \in X(k)$. Let $N^F(X, x)$ be the full subcategory of $N(X, x)$ whose objects consist of pointed torsors with finite local groups. This category is also filtered so we can write $\pi^F(X, x) := \varprojlim_{N^F(X,x)} G$. If $X$ is also proper and $k$ is perfect, then $\pi^F(X, x)$ is the Tannakian group of the full subcategory of the category of essentially finite vector bundles $\mathrm{Ess}(X)$ consisting of $F$-trivial bundles, i.e. vector bunldes which are trivial after pull back along some relative Frobenius $\phi_{(-t)} : X^{(-t)} \to X$ with $t \in \mathbb{N}$.

COROLLARY 2.1.2. *Let $X$ and $Y$ be two reduced connected proper schemes over a perfect field $k$. Let $x \in X(k)$, $y \in Y(k)$. Then the canonical map*

$$\pi^F(X \times_k Y, (x, y)) \to \pi^F(X, x) \times_k \pi^F(Y, y)$$

*is an isomorphism of $k$-group schemes.*

PROOF. We will use the obvious analogues of Theorem 2.0.4 to prove this theorem. Note that after replacing $\pi^N(X,x)$ by $\pi^F(X,x)$, "torsor" by "local torsor" (torsors whose groups are local), essentially finite vector bundle by $F$-trivial vector bundle, Theorem 1.0.23 and Theorem 2.0.4 are still true.

To prove this corollary we only need to show that the sequence

$$1 \to \pi^F(Y,y) \to \pi^F(X\times_k Y,(x,y)) \to \pi^F(X,x) \to 1$$

is exact. So we have to check that for any $G$-saturated local torsor $(P,G,p) \in N^F(X,x)$, $V := \pi_* O_P$ ($\pi : P \to X$ is the structure map) satisfies base change at $x$ and $f_*V$ is an $F$-trivial vector bundle.

Now suppose that $V$ is trivialized by

$$X^{(-t)} \times_k Y^{(-t)} = (X\times_k Y)^{(-t)} \to X\times_k Y.$$

Consider the following commutative diagram

$$\begin{tikzcd} Y \arrow[r, "y"] \arrow[d, equal] & X^{(-t)}\times_k Y \arrow[r, "p_1"] \arrow[d, "\phi_{(-t)}\times id"] & X^{(-t)} \arrow[d, "\phi_{(-t)}"] \\ Y \arrow[r, "y"] & X\times_k Y \arrow[r, "f"] & X \end{tikzcd}.$$

Let $W$ be the pull back of $V$ via $X^{(-t)}\times_k Y \to X\times_k Y$. Since $W$ has trivial fibres along the projection $p_2 : X^{(-t)}\times_k Y \to Y$ and $X^{(-t)}$ is proper separable and geometrically connected scheme, so there exists a vector bundle on $E$ on $Y$ such that $p_2^*E \cong W$, so $V$ has constant fibres along $f : X\times_k Y \to X$. Consequently base change is satisfied for $V$ along $f$ (at any point of $X$). On the other hand we have the following trivial cartesian diagram

$$\begin{tikzcd} X\times_k Y \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \arrow[d, "b"] \\ X \arrow[r, "a"] & \mathrm{Spec}\,(k) \end{tikzcd},$$

where $a$ and $b$ are structure maps. Because of base change we have $a^*b_*E \cong p_{1*}{p_2}^*E$. This implies $p_{1*}W = p_{1*}{p_2}^*E$ is a trivial vector bundle. But since $\phi_{(-t)} : X^{(-t)} \to X$ is faithfully flat, so we have a canonical isomorphism

$$p_{1*}W = p_{1*}(\phi_{(-t)}\times id)^*V \cong \phi^*_{(-t)} f_*V.$$

Thus $\phi^*_{(-t)} f_*V$ is a trivial vector bundle. By definition $f_*V$ is $F$-trivial. □

REMARKS 2.1.3. (1) Here we didn't assume $X$ or $Y$ is irreducible, this is because we have only used the sufficiency part of Theorem 2.0.4 in which only the citation of Theorem 1.0.23 used the irreduciblity. But the irreduciblity in Theorem 1.0.23 is only used for extending base change at one point to base change at all points. Since in the above proof we have already showed that base change holds for all $F$-trivial bundles at all points, so we don't need the irreduciblity.
(2) This corollary gives another way to see [**MS**][Proposition 2.1] which is the key point in the proof of the Künneth formula for Nori's fundamental group. But unfortunately, for

the full proof of of Künneth formula we have to use the same trick employed in [**MS**] to reduce the problem for $\pi^N$ to the problem for $\pi^F$. At the moment, I can not find any easy way to reduce the problem to $\pi^F$ using our language here.

## 3. The smooth case

We have seen in Chapter 1 §2.3 that there is a Tannakian description of Nori's fundamental group if we assume the base is smooth. In this section we will use this Tannakian description to give more criterion to determine the exactness of Nori's fundamental group. It turns out that the exactness of the homotopy sequence of Nori's fundamental group is equivalent to the constancy of the image of the following canonical map

$$\pi^N(X_s, x) \to \pi^N(X, x)$$

when $s$ varies in $S$. So as a consequence if all fibres $X_s$ have trivial Nori's fundamental group then the sequence is exact, this is just [**Nori**][Part I, Chapter II, Proposition 9]. We will also apply the criterion to the special case when the morphism $f : X \to S$ is a projection from a product (i.e.$X = T \times_k S$ and $f$ is the second projection), where the exactness becomes the Künneth formula. The criterion for homotopy sequence tells us that Künneth formula holds if and only if the canonical map of group schemes

$$\pi^N(T, t) \to \pi^N(T \times_k S, (t, s))$$

is constant when $s$ varies in $S$, but this is true if both $S$ and $T$ are proper smooth, this gives another proof of [**MS**][Proposition 2.1]. In the end we will show that if we take $S = \mathbb{A}^1$ and $T$ to be a supersingular elliptic curve then the canonical map

$$\pi^N(T, t) \to \pi^N(T \times_k S, (t, s))$$

may not be constant.

The notion of "fibrewise constancy" was first brought to my mind from Vikram Mehta through a discussion on Künneth formula. I thank him for this very helpful discussion.

Proposition 3.0.4. *If $S$ is a connected scheme smooth over a field $k = \bar{k}$ of positive characteristic, $f : X \to S$ is a smooth proper morphism with geometrically connected fibres, $s \in S(k)$, $x \in X(k)$ such that $f(x) = s$, then the homotopy sequence*

$$\pi^N(X_s, x) \to \pi^N(X, x) \to \pi^N(S, s) \to 1$$

*is exact if and only if for any $G$-saturated torsor $(P, G, p) \in N(X, x)$ with structure map $\pi : P \to X$, $\pi_* O_P$ satisfies base change at $s$ and there is a neighborhood $U$ of $s$ and an object $(W_i, \tau_i, i \geq 0) \in \mathrm{Strat}^{\mathrm{fin}}(U/k)$ which satisfies*

*(1) $W_0 = f_*\pi_*O_P|_U$;*

*(2) there is an imbedding $f^*(W_i, \tau_i, i \geq 0) \subseteq (V_i, \sigma_i, i \geq 0)|_{f^{-1}(U)}$ such that $f^*W_0 \to V_0|_{f^{-1}(U)}$ is the canonical map $f^*f_*\pi_*O_P \to \pi_*O_P$ restricting to $f^{-1}(U)$, where $(V_i, \sigma_i, i \geq 0) \in \mathrm{Strat}^{\mathrm{fin}}(X)$ is the stratified object corresponding to $(P, G)$.*

Proof. "$\Longrightarrow$" According to [**EPS**][Appendix A.1 (iii) (a)(b)], there is an object $(W_i, \sigma_i, i \geq 0) \in \mathrm{Strat}^{\mathrm{fin}}(S/k)$ which satisfies

(1) there is an imbedding $f^*(W_i, \tau_i, i \geq 0) \subseteq (V_i, \sigma_i, i \geq 0)$;
(2) if we restrict the imbedding $f^*W_0 \to V_0$ to $X_s$ then it gives the maximal trivial subbundle of $V_0|_{X_s}$.

In other words $f^*W_0|_{X_s}$ is equal to $H^0(X_s, V_0|_{X_s}) \otimes_k O_{X_s}$ as subbundles of $V_0|_{X_s}$ (see Chapter 2, 1.0.24). Thus the composition of maps

$$f^*W_0|_{X_s} \to f^*f_*V_0|_{X_s} \to H^0(X_s, V_0|_{X_s}) \otimes_k O_{X_s}$$

is an isomorphism. So each arrow is an isomorphism. Thus $f_*V_0$ has base change at $s$, so it is a vector bundle in a neighborhood $U$ of $s$, and the map $W_0 \to f_*V_0$ is an isomorphism on $U$. Hence the stratified sheaf $(W_i, \sigma_i, i \geq 0)|_{U/k}$ satisfies our conditions.

"$\Longleftarrow$" Let $G'$ be the Tannakian group of the Tannakian category generated by $(W_i, \tau_i, i \geq 0) \in \mathrm{Strat}^{\mathrm{fin}}(U)$ with fibre functor $s^*$. Then we get a surjection $G \to G'$ because of the embedding condition (2). Moreover one has the following commutative diagram

$$\begin{tikzcd}
\pi^N(X_s, x) \arrow[r] \arrow[d] & \pi^N(f^{-1}(U), x) \arrow[r] \arrow[d] & \pi^N(U, s) \arrow[r] \arrow[d] & 1 , \\
H \arrow[r] & G \arrow[r] & G' \arrow[r] & 1
\end{tikzcd}$$

where $H$ denotes the image of $\pi^N(X_s, x)$ in $G$. Now let $V$ be an object in $\mathsf{Rep}_k(G')$ corresponding to $(W_i, \tau_i, i \geq 0)$. Then by our condition (2) we have an imbedding $V \to k[G]$ in $\mathsf{Rep}_k(G)$. Again by condition (2) together with base change we have $V = k[G]^H$ as subojects of $k[G]$ in $\mathsf{Rep}_k(\pi^N(X_s, x))$ (see also Chapter 2, 1.0.24). But if we denote the kernel of $G \to G'$ by $N$ then we have $V \subseteq k[G]^N$ (since $V$ was in $\mathsf{Rep}_k(G')$). Thus the canonical inclusion $k[G]^N \subseteq k[G]^H$ is an isomorphism. This shows $H = N$. So the image $H$ is a normal subgroup of $G$. This together with base change at $s$ implies the exactness (Chapter 2, 1.0.23). $\square$

DEFINITION 3.0.5. Let $f : X \to S$ be a smooth map between two geometrically connected smooth schemes over a field $k = \bar{k}$ of positive characteristic with geometrically connected fibres, $x \in X(k)$, $s \in S(k)$ and $f(x) = s$. Let $(P, G, p) \in N(X, x)$ with structure map $\pi : P \to X$. In the following we will make precise the meaning that the image of $\pi^N(X_s, x) \to \pi^N(X, x) \to G$ is constant when $s$ varies in $S$.

If $s'$ is another $k$-point in $S$ and $x' \in f^{-1}(s'), p' \in \pi^{-1}(x')$, and if there is a $k$-linear tensor isomorphism between the fibre functors $x^*$ to $x'^*$ from $\mathrm{Strat}^{\mathrm{fin}}(X)$ to $\mathrm{Vec}_k$, then there are isomorphisms $\pi^N(X, x) \cong \pi^N(X, x')$, $\pi^N(S, s) \cong \pi^N(S, s')$ and $G \cong G$ which are induced by the chosen isomorphism between the fibre functors $x^*$ and $x'^*$ (because by definition we have $\pi^N(X, x) := \mathrm{Aut}^{\otimes}(x^*)$, $\pi^N(S, s) := \mathrm{Aut}^{\otimes}(s^*)$, and there is a canonical functor $\mathsf{Rep}_k(G) \to \mathrm{Strat}^{\mathrm{fin}}(X)$ induced from $(P, G, p)$ (see Chapter 1 §2.2)). So we get two commutative diagrams:

$$\begin{tikzcd} \pi^N(X_s,x) \arrow[r] & \pi^N(X,x) \arrow[r] \arrow[d, "\cong"] & \pi^N(S,s) \arrow[d, "\cong"] \\ \pi^N(X_{s'},x') \arrow[r] & \pi^N(X,x') \arrow[r] & \pi^N(S,s') \end{tikzcd} \qquad \begin{tikzcd} \pi^N(X,x) \arrow[r, "\cong"] \arrow[d] & \pi^N(X,x') \arrow[d] \\ G \arrow[r, "\cong"] & G \end{tikzcd}.$$

We say $f$ has fiberwise constant fundamental group at $s$ with respect to $(P,G,p)$ if $\exists$ a neighborhood $s \in U \subseteq S$ such that for any $s' \in U(k)$, there exists $x' \in f^{-1}(s'), p' \in \pi^{-1}(x')$ and an isomorphism between fibre functors $x^*$ to $x'^*$ such that the image of $\pi^N(X_s,x)$ and $\pi^N(X_{s'},x')$ in $G$ coincide under the induced automorphism $G \cong G$ as above.

THEOREM 3.0.6. *If $S$ is a connected scheme smooth over a field $k=\bar{k}$ of positive characteristic, $f: X \to S$ is a smooth proper morphism with geometrically connected fibres, $s \in S(k)$, $x \in X(k)$ such that $f(x)=s$, then the homotopy sequence*

$$\pi^N(X_s,x) \to \pi^N(X,x) \to \pi^N(S,s) \to 1$$

*is exact if and only if for any $G$-saturated torsor $(P,G,p) \in N(X,x)$ with structure map $\pi: P \to X$, $f$ has fiberwise constant fundamental group at $s$ with respect to $(P,G,p)$.*

PROOF. "$\Rightarrow$" Let $(P,G,p) \in N(X,x)$. Then by Proposition 3.0.4, there is a neighborhood $s \in U \subseteq S$ such that $\forall s' \in U$ the image of $\pi^N(X_{s'},x')$ and the image of the kernel of $\pi^N(X,x') \to \pi^N(S,s')$ are coincide in $G$. So if $x' \in f^{-1}(s')$ then any $k$-linear tensor isomorphism between $x^*$ and $x'^*$ would make the image of $\pi^N(X_{s'},x')$ and $\pi^N(X_s,x)$ coincide in $G$ under the automorphism $G \cong G$ which is induced by the isomorphism between $x^*$ and $x'^*$. So $f$ has fiberwise constant fundamental group at $s$ with respect to $(P,G,p)$.

"$\Leftarrow$" Suppose we have a $G$-saturated torsor $(P,G,p) \in N(X,x)$ with structure map $P \xrightarrow{\pi} X$. Let $V := \pi_* O_P$. then by Chapter 2, 1.0.24 the maximal trivial subbundle

$$H^0(X_s, V|_{X_s}) \subseteq V|_{X_s}$$

which corresponds via Tannakian duality to

$$k[G]^{\pi^N(X_s,x)} \to k[G]$$

is constant when $s$ varies in $S$. Hence $V$ as a vector bundle on $X$ satisfies base change at $s$, so by Proposition 3.0.4 we only have to show that $f_*V$ can be naturally extended to a $t$-stratified sheaf in a neighborhood of $s$ for some $t \geq 0$.

For simplicity we may assume the neighborhood $U$ in Definition 3.0.5 is $S$. Suppose the torsor $P$ corresponds to a $n$-stratified bundle $(V, V_1, V_2, \cdots, \sigma, \sigma_1, \sigma_2, \cdots)$, and suppose the étale quotient $P_{\text{ét}}$ of $P$ corresponds to the stratified bundle $(V_i, \sigma_i, i \geq 0)$ (see Chapter 1, Construction 2.4.3). Now look at the following commutative diagram:

$$\begin{tikzcd} X^{(-n)} \arrow[r, "v"] \arrow[d, "g"] & X \arrow[d, "f"] \\ S^{(-n)} \arrow[r, "u"] & S \end{tikzcd}.$$

One gets two maps

$$\begin{tikzcd} u^*f_*V \arrow[r, "\lambda"] & g_*v^*V \arrow[d, "\cong"] \\ u^*f_*V_0 \arrow[r, "\lambda_0"] & g_*v^*V_0 \end{tikzcd}$$

on $S^{(-n)}$ and a commutative diagram of $k$-group schemes:

$$\begin{tikzcd} \pi^N(X_t^{(-n)}, x_t) \arrow[r] \arrow[d] & \pi^N(X^{(-n)}, x_t) \arrow[r] \arrow[d] & \pi^N(S^{(-n)}, t) \arrow[r] \arrow[d] & 1 \\ \pi^N(X_t, x_t) \arrow[r] & \pi^N(X, x_t) \arrow[r] & \pi^N(S, t) \arrow[r] & 1 \end{tikzcd}$$

for any $t \in S(k) = S^{(-n)}(k)$. Using the lemma below and Chapter 2 Lemma 1.0.24, one can check that the pull back of $\lambda$ and $\lambda_0$ along the rational point $t$ are isomorphic to the maps $k[G]^{H_t} \hookrightarrow k[G]^{N_t}$ and $k[G]^{H'_t} \hookrightarrow k[G]^{N_t}$ respectively, where $H_t$ and $H'_t$ are the image of $\pi^N(X_t, x_t)$ in $G$ and $G_{\text{ét}}$ respectively and $N_t$ is the image of

$$\pi^N(X_t^{(-n)}, x_t) \to \pi^N(X^{(-n)}, x_t) \to G_{\text{ét}} \subseteq G.$$

Since the object $(V_i, \sigma_i, i \geq 0)$ corresponds to an étale Tannakian group and the homotopy sequence is exact for étale fundamental groups, there is an object $(W_i, \tau_i, i \geq 0) \in \mathrm{Strat}^{\mathrm{fin}}(S)$ such that $W_0 = f_*V_0$ and there is an embedding $f^*(W_i, \tau_i, i \geq 0) \subseteq (V_i, \sigma_i, i \geq 0)$ with the 0-th map $f^*W_0 \subseteq V_0$ equal to the canonical map $f^*f_*V_0 \to V_0$.

We claim that there is a unique arrow $\eta : u^*f_*V \to u^*f_*V_0$ which is compatible with $\lambda$ and $\lambda_0$ under the isomorphism $g_*v^*V \cong g_*v^*V_0$ and that there is a vector bundle $M_1$ on $S^{(1)}$ with an imbedding $M_1 \to W_1$ such that the pull back of this imbedding along $S^{(-n)} \to S^{(1)}$ is precisely $\eta$ under the identification $\tau_0 : \phi_0^*W_1 \cong W_0 = f_*V_0$.

The first statement is easy to show because $H'_t \subseteq H_t$ under the canonical section $H'_t = (H_t)_{\text{ét}} \subseteq H_t$. Thus we have $k[G]^{H_t} \subseteq k[G]^{H'_t} \subseteq k[G]$. Hence the composition

$$u^*f_*V \xrightarrow{\lambda} g_*v^*V \xrightarrow{\cong} g_*v^*V_0 \to g_*v^*V_0/u^*f_*V_0$$

is 0 on each fibre of $t \in S^{(-n)}$, so itself is 0. This implies our first statement.

For the second statement we first note that the faithful flatness of the relative Frobenius gives us the uniqueness of the map $M_1 \to W_1$ (if it exists). So we can construct $M_1$ and the imbedding locally can then glue them together. Hence we may assume our bundles $W_1, f_*V$ are all free. Now we choose a basis $\{u_1, u_2, \cdots, u_r\}$ in $W_1$ then it is naturally a basis in $u^*\phi_1^*W_1 \cong u^*W_0 = u^*f_*V_0$. We also fix a basis $\{v_1, v_2, \cdots, v_s\}$ in $f_*V$, then it naturally becomes a basis in $u^*f_*V$. The map $\eta$ will give us a $r \times s$ matrix $T$ with entries in $\Gamma(S^{(-n)}, O_{S^{(-n)}})$. But for all $t \in S(k) = S^{(-n)}(k)$, the fibres of $\eta$ on $t$ are all of the form $k[G]^{H_t} \subseteq k[G]^{H'_t}$ which is constant when $t$ varies in $S$ since $f$ has fibrewise constant fundamental group in $S$. Since $S$ is reduced, $T$ is a matrix with entries in $k$. Because $k$ is perfect, this tells us that we can descent $\eta$ to a morphism $M_1 \to W_1$. This proves our second statement. Let's denote the isomorphism $u^*\phi_1^*M_1 \cong u^*f_*V$ by $\delta$.

Now we use the same method employed in the proof of the second statement to descent $\eta$ to $M_i \to W_i$ for all $i \geq 1$. And since our relative Frobenius are all finite faithfully flat, we also get isomorphisms $\delta_i : \phi_i^* M_{i+1} \to M_i$ for $i \geq 1$. This makes $(f_*V, M_1, M_2, \cdots, \delta, \delta_1, \delta_2, \cdots)$ a $n$-stratified bundle. This stratified sheaf obviously satisfies all our requirements so our proof is completed. □

LEMMA 3.0.7. *If $S$ is a connected scheme smooth over a field $k = \bar{k}$ of positive characteristic, $f : X \to S$ is a smooth proper morphism with geometrically connected fibres, $s \in S(k)$, $x \in X(k)$ such that $f(x) = s$, then for any 0-stratified bundle $(V_i, \sigma_i, i \geq 0) \in \mathrm{Strat}^{\mathrm{fin}}(X, 0)$, $V_i$ satisfies base change at $s$ for all $i \geq 0$.*

PROOF. We just have to show base change for $V_0$ because $(V_1, V_2, \cdots, \sigma_1, \sigma_2, \cdots)$ is also in $\mathrm{Strat}^{\mathrm{fin}}(X, 0)$. Since the Tannakian group which corresponds to $(V_i, \sigma_i, i \geq 0)$ is étale and we have showed that the étale quotient of Nori's fundamental group has exact homotopy sequence, thus one can use the argument which we employed in the proof of Proposition 3.0.4 ( necessity part) to show that $V_0$ satisfies base change at $s$. □

COROLLARY 3.0.8. *Let $S$ be a connected scheme smooth over a perfect field $k$ of positive characteristic, $f : X \to S$ be a smooth proper morphism with geometrically connected fibres, $s \in S(k)$, $x \in X(k)$ such that $f(x) = s$. Assume further that $\exists$ a neighborhood $s \in U \subseteq S$ such that for $\forall t \in U(k)$ $\exists y \in X_t(k)$ satisfying $\pi^N(X_t, y) = 1$. Then $\pi^N(X, x) \to \pi^N(S, s)$ is an isomorphism.*

REMARK 3.0.9. This easily follows from the above theorem, and it yields a new proof of [**Nori**][Part I, Chapter II, Proposition 9] under the extra smoothness assumption for $S$.

LEMMA 3.0.10. *If $X$ and $Y$ are smooth (or proper reduced) connected schemes over an algebraically closed field $k$, then*

$$\pi^N(X \times_k Y, (x, y)) \to \pi^N(X, x) \times_k \pi^N(Y, y)$$

*is an isomorphism at some $x \in X(k)$ and $y \in Y(k)$ if and only if it is an isomorphism for all such $x$ and $y$.*

PROOF. Assume the Künneth formula holds for some $(x, y)$, by symmetry one only has to show for any $x_1 \in X(k)$ the Künneth formula still holds for $(x_1, y)$. Now chose an isomorphism of fibre functors $u : x^* \cong x_1^*$. This $u$ will induce a commutative diagram:

$$\begin{tikzcd}
\pi^N(X, x) \arrow[r] \arrow[d, "\cong"] & \pi^N(X \times_k Y, (x, y)) \arrow[r] \arrow[d, "\cong"] & \pi^N(Y, y) \arrow[d, "\cong"] \\
\pi^N(X, x_1) \arrow[r] & \pi^N(X \times_k Y, (x_1, y)) \arrow[r] & \pi^N(Y, y)
\end{tikzcd}$$

Since the first row is exact by Künneth formula for $(x, y)$ so the second row is also exact. Hence we also have Künneth formula for $(x_1, y)$. □

COROLLARY 3.0.11. *If $X$ and $Y$ are smooth connected schemes over an algebraically closed field $k$ and $Y$ is proper over $k$, then Künneth formula holds if and only if for any*

*$x, x_1 \in X(k)$ there is an isomorphism of additive tensor functors $(x \times id)^* \cong (x_1 \times id)^*$:*

$$\begin{tikzcd} \mathrm{Strat}^{\mathrm{fin}}(Y) \arrow[r, shift left, "(x\times id)^*"] \arrow[r, shift right, "(x_1\times id)^*"'] & \mathrm{Strat}^{\mathrm{fin}}(X \times_k Y) \end{tikzcd}$$

Proof. Let's first fix an isomorphism of fibre functors $u : x^* \cong x_1^*$, then it induces a diagram:

$$\begin{tikzcd} & \pi^N(X \times_k Y, (x,y)) \arrow[r] \arrow[d, "\widetilde{u}"] & \pi^N(X,x)) \times_k \pi^N(Y,y)) \arrow[d, "\widetilde{u}\times id"] \\ \pi^N(Y,y)) \arrow[ur, "x\times id"] \arrow[dr, "x_1\times id"'] & & \\ & \pi^N(X \times_k Y, (x_1,y)) \arrow[r] & \pi^N(X,x_1)) \times_k \pi^N(Y,y)) \end{tikzcd} .$$

The right side square in the diagram is always commutative while the left triangle is commutative if and only if $(x \times id)^*$ and $(x_1 \times id)^*$ are isomorphic as additive tensor functors.

Since $\pi^N(Y,y) \to \pi^N(X,x)) \times_k \pi^N(Y,y))$ is always of the form $e \times id$ regardless of what $x \in X(k)$ is, so dropping the mid-arrow $\widetilde{u}$, the diagram is commutative. But if Künneth formula holds then all the maps in the square of the above diagram are isomorphisms. Hence the left triangle must commute. Conversely if the left triangle commutes then using theorem 3.0.6 we conclude that Künneth formula holds. ☐

Definition 3.0.12. Let $X$ be a reduced connected scheme over a perfect field $k$ with a rational point $x \in X(k)$. Let $N^F(X,x)$ be the full subcategory of $N(X,x)$ whose objects consists of pointed torsors with finite local groups. We write $\pi^F(X,x) := \varprojlim_{N^F(X,x)} G$. If $X$ is also proper, we will define $\mathcal{C}^F(X)$ to be the full subcategory of the category of essentially finite vector bundles (which is denoted by $\mathrm{Ess}(X)$) consists of vector bunldes which are trivial after pull back along some relative Frobenius $\phi_t : X^{(-t)} \to X$. If $X$ is also smooth (may not be proper), we define $\mathcal{C}^F(X)$ to full subcategory of $\mathrm{Strat}^{\mathrm{fin}}(X)$ consists of $t$-stratified bundles $(E_i, \sigma_i, i \geq 0)$ with $E_i$ and $\sigma_i$ trivial (i.e. $E_i = \oplus O_{X^{(i)}}$ and $\sigma_i = id$) for all $i \geq 1$.

Lemma 3.0.13. *Let $X$ be a proper reduced (or smooth) connected scheme over a perfect field $k$ with a rational point $x \in X(k)$. Then $\pi^F(X,x)$ is the Tannakian group of $\mathcal{C}^F(X)$ with the fibre functor $x^*$.*

Proof. If $V \in \mathcal{C}^F(X)$ an object which is trivialized by $\phi_t : X^{(-t)} \to X$ with $t \in \mathbb{N}$. Then the full sub abelian tensor category of $\mathrm{Ess}(X)$ (or $\mathrm{Strat}^{\mathrm{fin}}(X)$) generated by $V$ with $x^*$ correspond to a finite group scheme $G$ over $k$. Thus we have an additive tensor functor

$$\lambda : \mathsf{Rep}_k(G) \to \mathrm{Ess}(X)$$

which 2-commutes with the fibre functors. This gives us a $G$-torsor $P$ which admits a rational point $p$ over $x$. But if we compose $\lambda$ with the canonical pull-back functor $\phi^* : \mathrm{Ess}(X) \to \mathrm{Ess}(X^{(i)})$ then the result is just the "free" functor, i.e. sending $W \in \mathsf{Rep}_k(G)$

to $W \otimes_k O_{X^{(i)}}$. Thus $P \to X$ is a trivial $G$-torsor after pull back to $X^{(i)}$. This means the group homomorphism $G^{(i)} \to G$ is trivial. This implies $G$ is a local group scheme. Hence $V$ is contained in the essential image of the imbedding

$$\mathsf{Rep}_k(\varprojlim_{N^F(X,x)}(G)) = \varinjlim_{N^F(X,x)} \mathsf{Rep}_k(G) \longrightarrow \mathrm{Ess}(X).$$

But the imbedding clearly factors $\mathcal{C}^F(X) \subseteq \mathrm{Ess}(X)$. Thus we have

$$\varinjlim_{N^F(X,x)} G \longrightarrow \mathcal{C}^F(X) \subseteq \mathrm{Ess}(X)$$

is an equivalence of categories. This concludes our proof. □

COROLLARY 3.0.14. *If $X$ and $Y$ are proper smooth connected schemes over a perfect field $k$, $x \in X(k)$ and $y \in Y(k)$, then the canonical map*

$$\pi^F(X \times_k Y, (x,y)) \to \pi^F(X,x) \times_k \pi^F(Y,y)$$

*is an isomorphism.*

PROOF. We may assume $k = \bar{k}$. One checks readily that 3.0.4-3.0.11 hold well if we replace $\pi^N$ by $\pi^F$ and arbitrary torsor $(P,G,p)$ by local torsor (i.e. torsor with local group scheme). So we only have to construct an isomorphism of functors between $(x \times id)^*$ and $(x' \times id)^*$

$$\mathcal{C}^F(Y) \longrightarrow \mathcal{C}^F(X \times_k Y)$$

for some $x' \in X(k)$. Suppose $E := (E_i, \sigma_i, i \geq 0) \in \mathcal{C}^F(X \times_k Y) \subseteq \mathrm{Strat}^{\mathrm{fin}}(X \times_k Y)$, then there is a nature number $t \geq 0$ such that the pull-back of $E$ along the relative Frobenius $X^{(-t)} \times_k Y^{(-t)} \to X \times_k Y$ is a trivial object. This implies the pull-back of $E$ along the map $\delta : X^{(-t)} \times_k Y \to X \times_k Y$ has trivial fibres over $Y$. Since $X^{(-t)}$ is also smooth proper connected there exists a canonical object $E' = (p_*E_i, p_*\sigma_i, i \geq 0) \in \mathcal{C}^F(Y)$ such that the pull-back of it along the projection $p : X^{(-t)} \times_k Y \to Y$ is canonically isomorphic to $E$. Thus we have $x^*E \cong E' \cong x'^*E$. One checks readily that this isomorphism does not depend on the choice of $t$ and that the isomorphism is functorial in $E$. This defines an isomorphism of functors. □

REMARK 3.0.15. This corollary gives another way to see [**MS**][Proposition 2.1] which is the key point in the proof of Künneth formula. But unfortunately, we have to put extra smoothness assumption. For the full proof of of Künneth formula we have to use the same trick employed in [**MS**] to reduce the problem for $\pi^N$ to the problem for $\pi^F$. At the monent, I can not find any easy way to reduce the problem to $\pi^F$ using our language here.

**3.1. A counterexample.** Now consider $k$ an algebraically closed field of characteristic 2, $X = \mathbb{A}^1_k$, $Y = E$ a supersingular elliptic curve. We want to find an object in $\mathcal{C}^F(\mathbb{A}^1_k \times_k E)$ such that the condition in Corollary 3.0.11 does not hold.

Suppose $\pi : P \to E$ is a non-trivial $\alpha_2$-tosor over $E$. Let $V := \pi_* O_P$. Let $\mathcal{L}$ be the cokernel of the structure map $O_E \to V$, then $\mathcal{L}$ is an essentially finite line bundle, so it has degree 0. If $\mathcal{L}$ was not $O_E$, then $H^1(E, \mathcal{L}^{-1}) = \mathrm{Ext}^1(O_X, \mathcal{L}^{-1}) \neq 0$, so by Riemann-Roch $h^0(E, \mathcal{L}^{-1}) = h^1(E, \mathcal{L}^{-1}) \neq 0$. But this implies $L^{-1}$ is $O_E$ which is impossible. Hence we have $\mathcal{L} \cong O_E$. This gives us an exact sequence

$$0 \to O_E \to V \to O_E \to 0.$$

We know that for $i = -1$, $P \to E$ has already became a trivial torsor after pulling back along the relative Frobenius $\phi_i : E^{(i)} \to E$. Thus after choosing a section $E^{(i)} \to P \times_E E^{(i)}$ we get an $E^{(i)}$-scheme isomorphism $P \times_E E^{(i)} \cong \alpha_2 \times_k E^{(i)}$ which gives us a trivialization $\delta : {\phi_i}^* V \cong O_{E^{(i)}} \oplus O_{E^{(i)}}$ making the diagram

$$\begin{tikzcd}
0 \arrow[r] & O_{E^{(i)}} \arrow[r] \arrow[d, equal] & {\phi_i}^* V \arrow[r] \arrow[d, "\delta"] & O_{E^{(i)}} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & O_{E^{(i)}} \arrow[r] & O_{E^{(i)}} \oplus O_{E^{(i)}} \arrow[r] & O_{E^{(i)}} \arrow[r] & 0
\end{tikzcd}$$

commutative. By Grothendieck's FPQC descent theory there is an essentially unique descent isomorphism $\varepsilon$ corresponding to $V$

$$\begin{tikzcd}
p_1^* {\phi_i}^* V \arrow[r, "\cong"] \arrow[d, "p_1^*\delta"] & p_2^* {\phi_i}^* V \arrow[d, "p_2^*\delta"] \\
O_{E^{(i)} \times_E E^{(i)}} \oplus O_{E^{(i)} \times_E E^{(i)}} \arrow[r, "\varepsilon"] & O_{E^{(i)} \times_E E^{(i)}} \oplus O_{E^{(i)} \times_E E^{(i)}}
\end{tikzcd},$$

where $p_1$ and $p_2$ are the two projections of $E^{(i)} \times_E E^{(i)}$. This $\varepsilon$ is expressible by a matrix in $GL_2(\Gamma(E^{(i)} \times_E E^{(i)}, O_{E^{(i)} \times_E E^{(i)}}))$

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}.$$

Since $P$ is not a trivial torsor $a \neq 0$.

Let $x$ be the indeterminate in $X = \mathbb{A}_k^1 = \operatorname{Spec}(k[x])$. Then the $2 \times 2$-matrix:

$$\begin{pmatrix} 1 & ax \\ 0 & 1 \end{pmatrix}$$

in $GL_2(\Gamma(X \times_k E^{(i)} \times_E E^{(i)}, O_{X \times_k E^{(i)} \times_E E^{(i)}}))$ determines an isomorphism

$$\varepsilon' : O_{X \times_k E^{(i)} \times_E E^{(i)}} \oplus O_{X \times_k E^{(i)} \times_E E^{(i)}} \cong O_{X \times_k E^{(i)} \times_E E^{(i)}} \oplus O_{X \times_k E^{(i)} \times_E E^{(i)}}.$$

One checks readily that the pair

$$(O_{X \times_k E^{(i)}} \oplus O_{X \times_k E^{(i)}}, \varepsilon')$$

gives us a descent data. This descent data will give us a rank 2 vector bundle $W$ on $X \times_k E$ with a trivialization $\xi : {\phi_i}^* W \cong O_{X \otimes_k E^{(i)}} \oplus O_{X \otimes_k E^{(i)}}$ on $X \times_k E^{(i)}$. Let $\lambda$ be the pull back of $\xi$ along the relative Frobenius $X^{(i)} \times_k E^{(i)} \to X \times_k E^{(i)}$. Then the pair $(W, \lambda)$ is a 1-stratified bundle over $X \times_k E/k$ (See Definition 3.0.12). If the Tannakian group corresponding to $(W, \lambda)$ is finite, then $(W, \lambda) \in \mathcal{C}^F(X \times_k E)$ (Definition 3.0.12).

It is clear that this $W$ does not have constant fibre along the projection $X \times_k E \to X$ (the fibre along $x = 0$ splits while the fibre along $x = 1$ does not). If it lies in $\mathcal{C}^F(X \times_k E)$, then it provides an example which does not satisfy the condition in Corollary 3.0.11. So we only have to prove $(W,\lambda) \in \mathcal{C}^F(X \times_k E)$.

Now we consider the tensor product $(W,\lambda)\otimes(W,\lambda)$. We see that $W \otimes_{O_E} W$ corresponds to the descent data

$$A := \begin{pmatrix} 1 & ax \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & ax \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & ax & ax & a^2x^2 \\ 0 & 1 & 0 & ax \\ 0 & 0 & 1 & ax \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

One the other hand we have $(W,\lambda) \oplus (W,\lambda)$, and $W \oplus W$ corresponds to the following descent data:

$$B := \begin{pmatrix} 1 & ax & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & ax \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

We will show that $(W,\lambda)\otimes(W,\lambda) \cong (W,\lambda)\oplus(W,\lambda)$, and hence $(W,\lambda)$ sits in our category $\mathcal{C}^F(X \times_k E)$.

We first come back to $V$. After fixing a section $E^{(i)} \to P$ over $E$, there are two ways to trivialize $V \otimes_{O_E} V$: one way is to pull it back along $\phi_i : E^{(i)} \to E$, then $\phi_i^* V \otimes_{O_{E^{(i)}}} \phi_i^* V$ is trivialized via the canonical trivialization $\phi_i^* V \cong O_{E^{(i)}} \oplus O_{E^{(i)}}$; the other way is to use the canonical isomorphism $P \times_k G \to P \times_X P$ to write $V \otimes_{O_E} V \cong V \oplus V$, then pull it back to $E^{(i)}$ and trivialize $V \oplus V$. We denote $\delta_1^*$ the first trivialization and $\delta_2^*$ the second trivialization. We want to know what is the relation between them.

Now let $P_1 = P_2 = P_3 = P$. Then the first trivialization of $V \otimes_{O_E} V$ can be reinterpreted as the following isomorphisms:

$$\delta_1^{-1} : P_1 \times_X P_2 \times_X P_3 \cong (P_1 \times_X P_2) \times_{P_1} (P_1 \times_X P_3) \cong (P_1 \times_k \alpha_2) \times_{P_1} (P_1 \times_k \alpha_2) \cong P_1 \times_k \alpha_2 \times_k \alpha_2.$$

This map sends $(p_1,p_2,p_3)$ to $(p_1,g_{12},g_{13})$ using funtorial viewpoint, where $p_1 g_{12} = p_2$ and $p_1 g_{13} = p_3$. The second trivialization of $V \otimes_{O_E} V$ comes from the following identifications:

$$\delta_2^{-1} : P_1 \times_X P_2 \times_X P_3 \cong P_1 \times_X P_2 \times_k \alpha_2 \cong P_1 \times_k \alpha_2 \times_k \alpha_2.$$

This map sends $(p_1,p_2,p_3)$ to $(p_1,g_{12},g_{23})$. Thus to identify our identifications we set an isomorphism $\theta : \alpha_2 \times_k \alpha_2 \to \alpha_2 \times_k \alpha_2$ sending $(x,y) \mapsto (x,xy)$. Thus we get a commutative diagram:

$$\begin{tikzcd} & P_1 \times_k \alpha_2 \times_k \alpha_2 . \\ P_1 \times_X P_2 \times_X P_3 \arrow[ur, "\delta_1^{-1}"] \arrow[dr, "\delta_2^{-1}"'] & \\ & P_1 \times_k \alpha_2 \times_k \alpha_2 \arrow[uu, "id\times\theta"'] \end{tikzcd}$$

Pulling the diagram back along the chosen section $E^{(i)} \to P_1$ we get our desired commutative diagram:

$$\begin{tikzcd}
 & O_{E^{(i)}} \oplus O_{E^{(i)}} \oplus O_{E^{(i)}} \oplus O_{E^{(i)}} \arrow[d, "\theta^* \times id"] \\
\phi_i^* V \otimes_{O_{E^{(i)}}} \phi_i^* V \arrow[ur, "\delta_1^*"] \arrow[r, "\delta_2^*"'] & O_{E^{(i)}} \oplus O_{E^{(i)}} \oplus O_{E^{(i)}} \oplus O_{E^{(i)}}
\end{tikzcd}$$

Now it's time to analyze the basis we have chosen for $\alpha_2 \times_k \alpha_2$ and hence give the matrix for $\theta^*$.

$$\theta^* : k[x]/x^2 \otimes_k k[y]/y^2 \to k[x]/x^2 \otimes_k k[y]/y^2$$

is determined by the association $x \mapsto x$ and $y \mapsto x \otimes 1 + 1 \otimes y$. To get the matrix

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a & a & a^2 \\ 0 & 1 & 0 & a \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

we have chosen the basis as $1 \otimes 1, 1 \otimes y, x \otimes 1, x \otimes y$. To get the second matrix

$$\begin{pmatrix} 1 & a & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

we have used the following basis $1 \otimes 1, x \otimes 1, 1 \otimes y, x \otimes y$. Since we have $\theta^*(1 \otimes 1) = 1 \otimes 1$, $\theta^*(1 \otimes y) = x \otimes 1 + 1 \otimes y$, $\theta^*(x \otimes 1) = x \otimes 1$, $\theta^*(x \otimes y) = x^2 \otimes 1 + x \otimes y = x \otimes y$. Thus the matrix of $\theta^*$ can be written as

$$C := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since the commutative diagram $(*)$:

$$\begin{tikzcd}
\phi_i^* V \otimes_{O_{E^{(i)}}} \phi_i^* V \arrow[r, "\delta_1^*"] \arrow[d, "\cong"'] \arrow[dr, dotted, "\delta_2^*"] & O_{E^{(i)}} \oplus O_{E^{(i)}} \oplus O_{E^{(i)}} \oplus O_{E^{(i)}} \arrow[d, "\theta^*"] \\
\phi_i^* V \oplus \phi_i^* V \arrow[r, "\cong"'] & O_{E^{(i)}} \oplus O_{E^{(i)}} \oplus O_{E^{(i)}} \oplus O_{E^{(i)}}
\end{tikzcd}$$

is compatible with the descent data we should have an equality of matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a & a & a^2 \\ 0 & 1 & 0 & a \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

That is, we should have

$$\begin{pmatrix} 1 & a & a & a^2 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & a \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a & a & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & a \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Thus, in conclusion, we have $a^2 = 0$, and in $GL_4(\Gamma(\mathbb{A}^1_k \times_k E^{(i)} \times_E E^{(i)}))$ the following equality:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & ax & ax & a^2x^2 \\ 0 & 1 & 0 & ax \\ 0 & 0 & 1 & ax \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & ax & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & ax \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

This tells us that we have obtained a commutative diagram of descent data

$$\begin{tikzcd}
\phi_i^* W \otimes_{O_{\mathbb{A}^1_k \times_k E^{(i)}}} \phi_i^* W \arrow[r, "\lambda\otimes\lambda"] \arrow[d, "\cong"'] & O_{\mathbb{A}^1_k \times_k E^{(i)}} \oplus O_{\mathbb{A}^1_k \times_k E^{(i)}} \oplus O_{\mathbb{A}^1_k \times_k E^{(i)}} \oplus O_{\mathbb{A}^1_k \times_k E^{(i)}} \arrow[d, "\theta^{**}"] \\
\phi_i^* W \oplus \phi_i^* W \arrow[r, "\lambda\oplus\lambda"'] & O_{\mathbb{A}^1_k \times_k E^{(i)}} \oplus O_{\mathbb{A}^1_k \times_k E^{(i)}} \oplus O_{\mathbb{A}^1_k \times_k E^{(i)}} \oplus O_{\mathbb{A}^1_k \times_k E^{(i)}}
\end{tikzcd}.$$

There is a unique isomorphism $W \otimes_{O_E} W \to W \oplus W$ corresponding to the map of descent data $\theta^{**}$. This isomorphism is also an isomorphism between $(W, \lambda) \otimes (W, \lambda)$ and $(W, \lambda) \oplus (W, \lambda)$ since the horizontal arrows in the above diagram are our trivializations.

Remark 3.1.1. Up to now the counterexample should have been finished, yet we find something more when we were playing around. This $W$ we have constructed is actually from an $\alpha_2$-torsor over $X \times_k E$.

One can see that the descent data

$$\begin{pmatrix} 1 & ax \\ 0 & 1 \end{pmatrix}$$

induces an $X \times_k E^{(i)} \times_E E^{(i)}$−automorphism of $\alpha_2 \times_k X \times_k E^{(i)} \times_E E^{(i)}$. That is to say the descent data is a descent data for affine schemes. Thus $W$ is a coherent sheaf of $O_{X\times_k E}$-algebra. Let $Q = \mathrm{Spec}\,(W)$, then the matrix $A$ is the descent data for $Q \times_{X\times_k E} Q$ and the matrix $B$ is the descent data for $Q \times_k \alpha_2$. Since the scheme isomorphism

$$id \times \theta : X \times_k E^{(i)} \times_k (\alpha_2 \times_k \alpha_2) \to X \times_k E^{(i)} \times_k (\alpha_2 \times_k \alpha_2)$$

corresponds to the map $\theta^{**}$, it is automatically a map of descent data for affine schemes. Thus we get an $X \times_k E$-isomorphism: $\rho : Q \times_k \alpha_2 \to Q \times_{X\times_k E} Q$. By FPQC-descent, the composition of $\rho$ with the second projection of $Q \times_{X\times_k E} Q$ defines an action of $\alpha_2$ and the composition of $\rho$ with the first projection of $Q \times_{X\times_k E} Q$ is the first projection of $Q \times_k \alpha_2$ (because they are the case after pulling back to $X \times_k E^{(i)}$). This tells us that $Q$ is an $\alpha_2$-torsor over $X \times_k E$.

REMARK 3.1.2. Hélène Esnault and Andre Chatzistamatiou pointed to us the following improvement of the above example. Thanks to their suggestion our counterexample may work for any characteristic $p > 0$. Now we consider the exact sequence of abelian sheaves in the flat topology

$$0 \to \alpha_p \to \mathbb{G}_a \xrightarrow{F} \mathbb{G}_a \to 0,$$

we then get a long exact sequence of abelian groups:

$$\cdots \to H^0_{fl}(-, \mathbb{G}_a) \to H^1_{fl}(-, \alpha_p) \to H^1_{fl}(-, \mathbb{G}_a) \xrightarrow{H^1(F)} H^1_{fl}(-, \mathbb{G}_a) \to \cdots$$

If we put $-$ to be an elliptic curve $E$ and if there map $H^1(F)$ has none trivial kernel (like in our case), then we can choose some $a \neq 0$ in the that kernel. If we put $-$ to be $\mathbb{A}^1_k \times_k E$ then

$$a \otimes x \in H^1_{fl}(\mathbb{A}^1_k \times_k E, \mathbb{G}_a) = H^1_{fl}(E, \mathbb{G}_a) \otimes_k k[x].$$

If we choose an element $b \in H^1_{fl}(\mathbb{A}^1_k \times_k E, \alpha_p)$ such that $b \mapsto a \otimes x$, then $b$ is an $\alpha_2$-torsor with non-constant fibres along the projection $\mathbb{A}^1_k \times_k E \to \mathbb{A}^1_k$. This can not happen if Künneth formula was true.

CHAPTER 3

# The Homotopy Sequence for the Algebraic Fundamental Group

Let $f : X \to S$ be a proper smooth morphism between two smooth proper connected schemes over $\mathbb{C}$, $x \in X(\mathbb{C})$, $s \in S(\mathbb{C})$, and $f(x) = s$. Then there is a long exact sequence

$$\cdots \pi_2^{\mathrm{top}}(S^{an}, s) \to \pi_1^{\mathrm{top}}(X_s^{an}, x) \to \pi_1^{\mathrm{top}}(X^{an}, x) \to \pi_1^{\mathrm{top}}(S^{an}, s) \to 1,$$

where $X^{an}$ ($S^{an}$ etc.) is the associated analytic space of the scheme $X$, and $\pi^{\mathrm{top}}$ is the topological fundamental group associated to the corresponding analytic topology. Since the the algebraic fundamental groups are the algebraic completions of the topological ones, so one should have a short exact sequence (since we don't have $\pi_2^{\mathrm{alg}}$ so we only have short exact sequence):

$$\pi^{\mathrm{alg}}(X_s, x) \to \pi^{\mathrm{alg}}(X, x) \to \pi^{\mathrm{alg}}(S, s) \to 1.$$

In the chapter we will prove the exactness for the algebraic fundamental groups over a field characteristic 0. For characteristic $p$, although we still don't have a proof yet, but we can prove a special case–the Künneth formula. This also gives some evidence for the exactness of the homotopy sequence in characteristic $p$.

## 1. The settings

**1.1. The general criterion.** In [**EPS**][Appendix Theorem A.1], Hélène Esnault, Phùng Hô Hai, Xiaotao Sun formulated a necessary sufficient condition for the exactness of Tannakian groups by looking at their corresponding tensor functors. Since the algebraic fundamental group is defined by Tannakian duality, so to prove the exactness we have to check the condition holds in our settings. For the convenience of the reader we rewrite the condition in the following theorem. For the proof one has to look at that article.

THEOREM 1.1.1. ([**EPS**][Appendix Theorem A.1]) *Let $L \xrightarrow{q} G \xrightarrow{p} A$ be a sequence of homomorphisms of affine group schemes over a field $k$. It induces a sequence of functors:*

$$\mathsf{Rep}_k(A) \xrightarrow{p^*} \mathsf{Rep}_k(G) \xrightarrow{q^*} \mathsf{Rep}_k(L),$$

*where $\mathsf{Rep}_k(-)$ denotes the category of finite dimensional representations of $-$ over $k$. Then we have*

1. *The group homomorphism $p : G \to A$ is faithfully flat if and only if $p^*\mathsf{Rep}_k(A)$ is a full subcategory of $\mathsf{Rep}_k(G)$ and closed under taking subquotients.*
2. *The group homomorphism $q : L \to G$ is a closed immersion if and only if any object of $\mathsf{Rep}_k(L)$ is a subquotient of an object of the form $q^*(V)$ for some $V \in \mathsf{Rep}_k(G)$.*
3. *Assume that $q$ is a closed immersion and that $p$ is faithfully flat. Then the sequence $L \xrightarrow{q} G \xrightarrow{p} A$ is exact if and only if the following conditions are fulfilled:*

(a) *For an object $V \in \mathsf{Rep}_k(G)$, $q^*V \in \mathsf{Rep}_k(L)$ is trivial if and only if $V \cong p^*U$ for some $U \in \mathsf{Rep}_k(A)$*
(b) *Let $W_0$ be the maximal trivial subobject of $q^*V$ in $\mathsf{Rep}_k(L)$. Then there exists $V_0 \subseteq V$ in $\mathsf{Rep}_k(G)$, such that $q^*V_0 \cong W_0$.*
(c) *Any $W$ in $\mathsf{Rep}_k(L)$ is embeddable in $q^*V$ for some $V \in \mathsf{Rep}_k(G)$.*

**1.2. The settings.** Let $f : X \to S$ be a smooth proper morphism with geometrically connected fibres between two smooth connected schemes of finite type over a field $k$, $s \in S(k)$ be a rational point, $X_s$ be the fibre, $x \in X(k)$ be a rational point lying above $s$, then by the functoriality of the algebraic fundamental group we get a sequence of affine group schemes

$$\pi^{\mathrm{alg}}(X_s, x) \to \pi^{\mathrm{alg}}(X, x) \to \pi^{\mathrm{alg}}(S, s) \to 1,$$

which is called the homotopy sequence. We will show that the sequence is exact if $k$ has characteristic 0 by checking the conditions provided in the above theorem.

## 2. The homotopy exact sequence in characteristic 0

In this section $k$ is always a field of characteristic 0. In this case the category $\mathsf{Mod}_c(D_{X/k})$ is the same as the category of vector bundles with flat connections, so in the following we will work purely in the category of vector bundles with flat connections and still use $\mathsf{Mod}_c(D_{X/k})$ to denote this category.

### 2.1. The conditions (a), (b) and the surjectivity.

THEOREM 2.1.1. *Notations and assumptions being as in §1.2, then the homotopy sequence*

$$\pi^{\mathrm{alg}}(X_s, x) \to \pi^{\mathrm{alg}}(X, x) \to \pi^{\mathrm{alg}}(S, s) \to 1$$

*is a complex, and the arrow $\pi^{\mathrm{alg}}(X, x) \to \pi^{\mathrm{alg}}(S, s)$ is surjective.*

PROOF. Since $s \in S$ is a rational point, we know that any object in $\mathsf{Mod}_c(D_{S/k})$ is trivial after pulling back to $\mathsf{Mod}_c(D_{X_s/k})$, thus the sequence is a complex. To see the right arrow is surjective, one has to show that the functor

$$f^* : \mathsf{Mod}_c(D_{S/k}) \to \mathsf{Mod}_c(D_{X/k})$$

is fully faithful and stable under taking subquotient.

The fact that $f^*$ is fully faithful follows readily from the projection formula, so we only have to show that it is stable under taking subquotient. Suppose we have an object $(E, \nabla_E) \in \mathsf{Mod}_c(D_{S/k})$, and a subobject

$$(F, \nabla_F) \hookrightarrow f^*(E, \nabla_E).$$

Then $f_*F$ is a locally free sheaf of rank equal to that of $F$, $f^*f_*F \to F$ is an isomorphism, and the natural map $f_*F \to E$ imbeds $f_*F$ as a subbundle of $E$ (locally split). This can be seen in the following way.

First of all, for any $t \in S$ $F|_{X_t}$ is a free $O_{X_t}$-module. This is because $f^*(E, \nabla_E)|_{X_t/\kappa(t)}$ is a trivial object in $\mathsf{Mod}_c(D_{X_t/\kappa(t)})$, but

$$(F, \nabla_F)|_{X_t/\kappa(t)} \subseteq f^*(E, \nabla_E)|_{X_t/\kappa(t)},$$

thus $(F, \nabla_F)|_{X_t/\kappa(t)}$ is also a trivial object, so $F|_{X_t}$ is a free $O_{X_t}$-module. This tells us $f_*F$ satisfies base change for any $t \in S$, hence is a vector bundle. Then the canonical map $f^*f_*F \to F$ is an isomorphism over all the fibres of $t \in S$, so itself is an isomorphism. This finishes the proof of the above claim.

Now from the connection $\nabla_F$, we get a map:

$$f_*F \to f_*(F \otimes_{O_X} \Omega^1_{X/k}) \cong f_*(f^*f_*F \otimes_{O_X} \Omega^1_{X/k}) \cong f_*F \otimes_{O_S} f_*\Omega^1_{X/k}.$$

Since $f : X \to S$ is smooth, the exact sequence

$$0 \to f^*\Omega^1_{S/k} \to \Omega^1_{X/k} \to \Omega^1_{X/S} \to 0$$

locally splits. Hence we have an induced injection

$$f^*(E/f_*F) \otimes_{O_X} f^*\Omega^1_{S/k} \hookrightarrow f^*(E/f_*F) \otimes_{O_X} \Omega^1_{X/k},$$

which is just

$$E/f_*F \otimes_{O_S} \Omega^1_{S/k} \hookrightarrow E/f_*F \otimes_{O_S} f_*\Omega^1_{X/k}$$

by applying $f_*$ and the projection formula. Now look at the following commutative diagramme with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & f_*F \otimes_{O_S} \Omega^1_{S/k} \arrow[r] \arrow[d] & E \otimes_{O_S} \Omega^1_{S/k} \arrow[r] \arrow[d] & E/f_*F \otimes_{O_S} \Omega^1_{S/k} \arrow[r] \arrow[d] & 0 . \\
0 \arrow[r] & f_*F \otimes_{O_S} f_*\Omega^1_{X/k} \arrow[r] & E \otimes_{O_S} f_*\Omega^1_{X/k} \arrow[r] & E/f_*F \otimes_{O_S} f_*\Omega^1_{X/k} \arrow[r] & 0
\end{tikzcd}$$

Since $f_*F$ maps to $f_*F \otimes_{O_S} f_*\Omega^1_{X/k}$, its image in $E/f_*F \otimes_{O_S} f_*\Omega^1_{X/k}$ is trivial. Because $E/f_*F \otimes_{O_S} \Omega^1_{S/k} \hookrightarrow E/f_*F \otimes_{O_S} f_*\Omega^1_{X/k}$ is injective, $f_*F \to E \otimes_{O_S} \Omega^1_{S/k}$ factors through $f_*F \otimes_{O_S} \Omega^1_{S/k}$. This proves that $f_*F \subseteq E$ is equipped with a flat connection $f_*\nabla_F$ which makes $(f_*F, f_*\nabla_F)$ a subobject of $(E, \nabla_E)$. Clearly $f^*(f_*F, f_*\nabla_F) \cong (F, \nabla_F)$ as subobjects of $f^*(E, \nabla_E)$. This finishes the proof. □

Corollary 2.1.2. *Notations and assumptions being as in §1.2, then for any object* $(E, \nabla_E) \in \mathsf{Mod}_c(D_{X/k})$, *the natural map*

$$\phi : f^*H^0_{DM}(X/S, (E, \nabla_E)) = f^*f_*E^{\nabla_{X/S}} \to E$$

*is a horizontal with respect to the Gauss-Manin connection on the left (i.e. a morphism in* $\mathsf{Mod}_c(D_{S/k})$*). Furthermore this map is injective and imbeds* $f^*H^0_{DM}(X/S, (E, \nabla_E))$ *as the maximal subobject of* $(E, \nabla_E)$ *coming from* $S/k$ *in the following sense:*

*If* $(M, \nabla_M) \subseteq (E, \nabla_E) \in \mathsf{Mod}_c(D_{X/k})$ *such that* $(M, \nabla_M) = f^*(N, \nabla_N)$ *for some* $(N, \nabla_N) \in \mathsf{Mod}_c(D_{S/k})$*, then the imbedding* $(M, \nabla_M) \subseteq (E, \nabla_E)$ *factors through* $\phi$.

Proof. The fact that $\phi$ is horizontal is from the definition of the Gauss-Manin connection. To show that it is injective one considers the kernel $(K, \nabla_K)$ of the map. One has:

$$0 \to (K, \nabla_K) \to f^*H^0_{DM}(X/S, (E, \nabla_E)) \xrightarrow{\phi} (E, \nabla_E)$$

is exact. Since the functor $H^0_{DM}(X/S, -)$ is left exact and $H^0_{DM}(X/S, \phi)$ is an isomorphism, we have $H^0_{DM}(X/S, (K, \nabla_K)) = 0$. But by the theorem above, one has $(K', \nabla_{K'}) \in$

$\mathsf{Mod}_c(D_{S/k})$ such that $f^*(K', \nabla_{K'}) = (K, \nabla_K)$. Thus as sheaves on $S$, one has

$$0 = H^0_{DM}(X/S, (K, \nabla_K)) = H^0_{DM}(X/S, f^*(K', \nabla_{K'})) \cong K'.$$

This shows that $K = 0$. Thus $\phi$ is injective.

Suppose $(M, \nabla_M) \subseteq (E, \nabla_E) \in \mathsf{Mod}_c(D_{X/k})$ such that $(M, \nabla_M) = f^*(N, \nabla_N)$ for some $(N, \nabla_N) \in \mathsf{Mod}_c(D_{S/k})$, then

$$N \cong f_* M^{\nabla_{X/S}} \hookrightarrow f_* E^{\nabla_{X/S}} = H^0_{DM}(X/S, (E, \nabla_E)),$$

this shows $M \hookrightarrow E$ factors through $\phi$. $\square$

THEOREM 2.1.3. *Notations and assumptions being as in §1.2, then for any* $(E, \nabla_E) \in \mathsf{Mod}_c(D_{X/k})$ *the subobject*

$$(F, \nabla_F) := f^* H^0_{DM}(X/S, (E, \nabla_E)) \hookrightarrow (E, \nabla_E)$$

*has the restriction* $(F, \nabla_F)|_{X_s/k} \hookrightarrow (E, \nabla_E)|_{X_s/k}$ *which gives the maximal trivial subobject of* $(E, \nabla_E)|_{X_s/k}$. *So in particular, our condition (a) and (b) are satisfied.*

PROOF. Since the maximal trivial subobject of $(E, \nabla_E)|_{X_s/k}$ is precisely

$$f^* H^0_{DM}(X_s/k, (E, \nabla_E)|_{X_s/k}) \hookrightarrow (E, \nabla_E)|_{X_s/k},$$

so the above theorem is just the base theorem for the Gauss-Manin Connection which was proved in [**Katz**][Section 8]. $\square$

**2.2. The condition (c) for a generic geometric point.** Now we come to check the condition (c) in our general criterion. Since we are not going to show the injectivity of the very left arrow, the condition (c) in our situation reads:

For $\forall (E, \nabla_E) \in \mathsf{Mod}_c(D_{X/k})$ and any quotient $(E, \nabla_E)|_{X_s/k} \twoheadrightarrow (F', \nabla_{F'}) \in \mathsf{Mod}_c(D_{X_s/k})$, $\exists (F, \nabla_F) \in \mathsf{Mod}_c(D_{X/k})$ and an imbedding $(F', \nabla_{F'}) \hookrightarrow (F, \nabla_F)|_{X_s/k} \in \mathsf{Mod}_c(D_{X_s/k})$. Or equivalently, one can say (by taking dual) for $\forall (E, \nabla_E) \in \mathsf{Mod}_c(D_{X/k})$ and any subobject $(E, \nabla_E)|_{X_s/k} \hookrightarrow (F', \nabla_{F'}) \in \mathsf{Mod}_c(D_{X_s/k})$, $\exists (F, \nabla_F) \in \mathsf{Mod}_c(D_{X/k})$ and a surjection $(F', \nabla_{F'}) \twoheadrightarrow (F, \nabla_F)|_{X_s/k} \in \mathsf{Mod}_c(D_{X_s/k})$.

This condition here is quite difficult to check, but since (a) and (b) are satisfied now it is equivalent to the exactness of the homotopy sequence. We will first prove this condition in a special case (for a generic geometric point) then we will show that if in this special case our condition is OK then the homotopy sequence is exact in general. Next we will place the settings for the generic geometric point.

**The Setup of §3.2:** Let $f_0 : X_0 \to S_0$ be a smooth morphism between smooth geometrically connected schemes of finite type over a field $k_0$ of characteristic 0. Let $k$ be an algebraic extension of $\kappa(S_0)$, $S := S_0 \times_{k_0} k$, $X := X_0 \times_{k_0} k$, $f := f_0 \times_{k_0} k$. Now we get a $k$-rational point $s \in S$ which corresponds to the generic point of $S_0$. Let $X_s$ be the fibre of $s \in S(k)$, $x \in X(k)$ such that $f(x) = s$.

PROPOSITION 2.2.1. *If* $(E, \nabla_E) \in \mathsf{Mod}_c(D_{X/k})$, $(F', \nabla_{F'}) \subseteq (E, \nabla_E)|_{X_s/k} \in \mathsf{Mod}_c(D_{X_s/k})$, *then* $\exists$ *a non-trivial Zariski open* $U_0 \subseteq S_0$ *and an object* $(F_0, \nabla_{F_0}) \in \mathsf{Mod}_c(D_{f_0^{-1}(U_0)/k_0})$ *with a surjection* $(F_0, \nabla_{F_0})|_{X_s/k} \twoheadrightarrow (F', \nabla_{F'})$.

PROOF. According to Lemma 2.2.2 below, we have a non-trivial Zariski open $U_0 \subseteq S_0$ and a finite étale covering $T_0 \to U_0$ with $\kappa(T_0) \subseteq k$ such that $(E, \nabla_E) \in \mathsf{Mod}_c(D_{X/k})$ is defined over $\mathsf{Mod}_c(D_{X_0 \times_{k_0} T_0/T_0})$. We may assume $U_0 = S_0$ and let $(E_0, \nabla_{E_0}) \in \mathsf{Mod}_c(D_{X_0 \times_{k_0} T_0/T_0})$ be the object such that $\rho^*(E_0, \nabla_{E_0}) \cong (E, \nabla_E)$ where $\rho : X = X_0 \times_{k_0} k \to X_0 \times_{k_0} T_0$. Let $\alpha : T_0 \hookrightarrow S_0 \times_{k_0} T_0$ be the graph of $T_0 \to S_0$ and $\beta : X_{0T_0} \hookrightarrow X_0 \times_{k_0} T_0$ be the pull back of the graph:

$$\begin{tikzcd} X_{0T_0} \arrow[r, "\beta"] \arrow[d] & X_0 \times_{k_0} T_0 \arrow[d] \\ T_0 \arrow[r, "\alpha"] & S_0 \times_{k_0} T_0 \end{tikzcd} .$$

Then the pull back $\beta^*(E_0, \nabla_{E_0}) \in \mathsf{Mod}_c(D_{X_{0T_0}/T_0})$ is actually defined over $\mathsf{Mod}_c(D_{X_{0T_0}/k_0})$. In fact, we have the following commutative diagram:

$$\begin{tikzcd} X_{0T_0} \arrow[r, "\beta"] \arrow[d] & X_0 \times_{k_0} T_0 \arrow[r, "p"] \arrow[d] & X_0 \arrow[d] \\ T_0 \arrow[r, equal] & T_0 \arrow[r] & k_0 \end{tikzcd} .$$

Thus we have maps

$$\beta^*\Omega^1_{X_0 \times_{k_0} T_0/T_0} \cong \beta^* p^* \Omega^1_{X_0/k_0} \to \Omega^1_{X_{0T_0}/k_0}.$$

Note that the last arrow in the above sequence is actually coming from the following commutative diagramme:

$$\begin{tikzcd} X_{0,T_0} \arrow[r, "p\circ\beta"] \arrow[d] & X_0 \arrow[d] \\ \mathrm{Spec}\,(k_0) \arrow[r, equal] & \mathrm{Spec}\,(k_0) \end{tikzcd} .$$

This indeed extends our connection

$$\nabla_{E_0} : E_0 \to E_0 \otimes_{O_{X_0 \times_{S_0} T_0}} \Omega^1_{X_0 \times_{k_0} T_0/T_0} \cong E_0 \otimes_{O_{X_0 \times_{S_0} T_0}} p^*\Omega^1_{X_0/k_0}$$

to the connection

$$\beta^*\nabla_{E_0} : \beta^* E_0 \to \beta^* E_0 \otimes_{O_{X_{0T_0}}} \Omega^1_{X_{0T_0}/k_0}.$$

Let $\lambda : X_{0T_0} \to X_{0S_0} \cong X_0$. Since $T_0 \to S_0$ is finite étale, we have $\lambda_*\beta^*(E_0, \nabla_{E_0}) \in \mathsf{Mod}_c(D_{X_0/k_0})$ and a surjection

$$\lambda^*\lambda_*\beta^*(E_0, \nabla_{E_0}) \twoheadrightarrow \beta^*(E_0, \nabla_{E_0}).$$

From the Cartesian diagrams

$$\begin{tikzcd} X_s \arrow[r, "\iota"] \arrow[d] & X_{0T_0} \arrow[r, "\beta"] \arrow[d] & X_0 \times_{k_0} T_0 \arrow[r] \arrow[d] & X_0 \arrow[d] \\ k \arrow[r] & T_0 \arrow[r] & S_0 \times_{k_0} T_0 \arrow[r] & S_0 \end{tikzcd}$$

we know that if we pull back $\beta^*(E_0, \nabla_{E_0})$ along $\iota$ then we get $(E, \nabla_E)|_{X_s/k}$. Now let $(F'', \nabla_{F''})$ be the inverse image of $(F', \nabla_{F'})$ under the map

$$\lambda^*\lambda_*\beta^*(E_0, \nabla_{E_0})|_{X_s/k} \twoheadrightarrow \beta^*(E_0, \nabla_{E_0})|_{X_s/k} = (E, \nabla_E)|_{X_s/k}.$$

According to our lemma 2.2.3 below, there exists non-trivial Zariski open $U_0 \subseteq S_0$ and $(F_0, \nabla_{F_0}) \in \mathsf{Mod}_c(D_{f_0^{-1}(U_0)/k_0})$ with a surjection $(F_0, \nabla_{F_0})|_{X_s/k} \twoheadrightarrow (F'', \nabla_{F''}) \twoheadrightarrow (F', \nabla_{F'}) \in \mathsf{Mod}_c(D_{X_s/k})$. This completes the proof. $\square$

LEMMA 2.2.2. (The notations and conventions in this lemma are independent) *Let $f : X \to S$ be a smooth morphism between two integral noetherian schemes. Let $s \in S$ be the generic point, $\kappa(s) \subseteq k$ be a separable algebraic extension of fields, $X_k$ be the generic fibre (corresponding to* Spec $(k) \hookrightarrow S$*). Then for any object $(F, \nabla_F) \in \mathsf{Mod}_c(D_{X_k/k})$ with $F$ a vector bundle, there exists a non-empty open subset $U \subseteq S$, an integral finite étale covering $T \to U$ and an object $(E, \nabla_E) \in \mathsf{Mod}_c(D_{X\times_S T/T})$ which satisfy (1) the function field of $T$ is contained in $k$; (2) $(F, \nabla_F) \cong (E, \nabla_E)_{X_k/k}$.*

PROOF. Let $\phi : X_k \to X$ be the canonical imbedding of the generic fibre and assume $S = \operatorname{Spec}(R)$. Then we get a surjection $\phi^*\phi_*F \twoheadrightarrow F$. Since $\phi_*F$ is the union of its coherent subsheaves, we find a coherent subsheaf $M$ of $\phi_*F$ with a surjection $\phi^*M \twoheadrightarrow F$. Suppose $N \subseteq \phi^*M$ is the kernel of $\phi^*M \twoheadrightarrow F$. It is coherent since $X$ is noetherian. Then we can collect finitely many elements $\{x_0, \cdots, x_n\}$ in $k$ which are integral over $R$ and a non-zero element $f \in R$ such that $N$ is defined over $R_1 := R_f[x_0, \cdots, x_n]$. Thus $F$ is defined over $R_1$. Let's say $E_1$ is a coherent sheaf on $X \times_R R_1$ such that $\rho_1^*E_1 \cong F$, where $\rho_1 : X_k = X \times_R k \to X \times_R R_1$. Since the problem is local for $S$, and $F$ is locally free, we may assume $E_1$ is locally free. Then the map

$$E_1 \otimes_{O_{X\times_R R_1}} \Omega^1_{X\times_R R_1/R_1} \to \rho_{1*}\rho_1^*(E_1 \otimes_{O_{X\times_R R_1}} \Omega^1_{X\times_R R_1/R_1})$$

is injective. Since the $k-$linear map

$$\nabla_F : F \to F \otimes_{O_{X_k}} \Omega^1_{X_k/k}$$

can be seen as a map

$$\rho_1^*E_1 \to \rho_1^*(E_1 \otimes_{O_{X\times_R R_1}} \Omega^1_{X\times_R R_1/R_1}),$$

we can collect finite many elements $\{y_0, \cdots, y_n\}$ in $k$ which are integral over $R$ and a non-zero element $g \in R$ such that $\nabla_F$ is defined over $R_2 = (R_1)_g[y_0, \cdots, y_n]$ and is still a flat connection. Thus we have found $T_2 := \operatorname{Spec} R_2$ and $(E_2, \nabla_{E_2}) \in FConn(X \times_S T_2/T_2)$ such that $\rho_2^*(E_2, \nabla_{E_2}) \cong (F, \nabla_F)$ (where $\rho_2 : X \times_S k \to X \times_S T_2$) and the generic point of $T_2$ is a finite field extension of $\kappa(s)$. Now the map $T_2 \to S$ which is finite onto its image is étale at the generic point of $T_2$, thus we get a non-empty open sub $T$ of $T_2$ such that $T$ is finite étale over some non-empty open $U$ of $S$. This is precisely what we want. $\square$

LEMMA 2.2.3. *For any object $(E, \nabla_E) \in \mathsf{Mod}_c(D_{X_0/k_0})$ and any imbedding $(F', \nabla_{F'}) \hookrightarrow (E, \nabla_E)|_{X_s/k} \in \mathsf{Mod}_c(D_{X_s/k})$ there is a non-empty open $U_0 \subseteq S_0$ and an object $(F, \nabla_F) \in \mathsf{Mod}_c(D_{f^{-1}(U_0)/k_0})$ which admits a surjection*

$$(F, \nabla_F)|_{X_s/k} \twoheadrightarrow (F', \nabla_{F'}) \hookrightarrow (E, \nabla_E)|_{X_s/k} \in \mathsf{Mod}_c(D_{X_s/k}).$$

Proof. First suppose $\kappa(S_0) \subseteq k$ is a trivial extension. Let $r := \dim_{O_{X_s}}(F')$. According to [**EP**][Theorem 5.10] we have a subobject $(M, \nabla_M) \subseteq (E, \nabla_E)|_{X_s/k_0} \in \mathsf{Mod}_c(D_{X_s/k_0})$ with a surjection $(M, \nabla_M)|_{X_s/k} \twoheadrightarrow \det(F', \nabla_{F'})$. If we set $(F_1, \nabla_{F_1}) := (M, \nabla_M) \otimes_{O_{X_s}} (\wedge^{r-1}(E, \nabla_E)|_{X_s/k_0})^\vee$, then it is a subobject

$$(F_1, \nabla_{F_1}) \subseteq (E, \nabla_E)|_{X_s/k_0} \otimes_{O_{X_s}} (\wedge^{r-1}(E, \nabla_E)|_{X_s/k_0})^\vee \in \mathsf{Mod}_c(D_{X_s/k_0})$$

with a surjection

$$(F_1, \nabla_{F_1})|_{X_s/k} \twoheadrightarrow (F', \nabla_{F'}) \cong \det(F', \nabla_{F'}) \otimes_{O_{X_s}} (\wedge^{r-1}(F', \nabla_{F'}))^\vee.$$

Let $u : X_s \to X_0$ be the canonical imbedding, then we take the inverse image of $u_*F_1$ under the canonical map

$$E \otimes_{O_X} (\wedge^{r-1}E)^\vee \to u_*u^*(E \otimes_{O_X} (\wedge^{r-1}E)^\vee)$$

and denote it by $F_2$. One can check there is a non-empty open subscheme $U_0 \subseteq S_0$ so that $F_2$ is equipped with a flat connection on $f^{-1}(U_0)/k_0$ and becomes a subobject

$$(F_2, \nabla_{F_2}) \subseteq ((E, \nabla_E) \otimes_{O_{X_0}} (\wedge^{r-1}(E, \nabla_E))^\vee)|_{f^{-1}(U_0)/k_0} \in \mathsf{Mod}_c(D_{f^{-1}(U_0)/k_0})$$

which satisfies $(F_2, \nabla_{F_2})|_{X_s/k_0} \cong (F_1, \nabla_{F_1})$. This finishes the special case.

Now suppose $\kappa(S_0) \subseteq k$ is a non-trivial extension. It is clear that the map $(F', \nabla_{F'}) \hookrightarrow (E, \nabla_E)|_{X_s/k}$ is defined over $\mathsf{Mod}_c(D_{X_{k'}/k'})$ where $k'$ is a finite extension of $\kappa(S_0)$ and $X_{k'} := X_0 \times_{S_0} k'$. Thus we may assume $k/\kappa(S_0)$ is finite. Then the map $\alpha : X_s \to X_0 \times_{S_0} \kappa(S_0)$ is finite étale. So we get a surjection

$$\alpha^*\alpha_*(F', \nabla_{F'}) \twoheadrightarrow (F', \nabla_{F'}) \in \mathsf{Mod}_c(D_{X_s/k})$$

and an imbedding

$$\alpha_*(F', \nabla_{F'}) \hookrightarrow \alpha_*((E, \nabla_E)|_{X_s/k}) \in \mathsf{Mod}_c(D_{X_0 \times_{S_0} \kappa(S_0)/\kappa(S_0)}).$$

Thus it is enough to show that $\alpha_*((E, \nabla_E)|_{X_s/k})$ is defined in $\mathsf{Mod}_c(D_{f^{-1}(U_0)/k_0})$ with $U_0 \subseteq S_0$ non-trivial Zariski open, since then we can apply the special case we discussed above to get a surjection on $\alpha_*(F', \nabla_{F'})$ from some object in $\mathsf{Mod}_c(D_{f^{-1}(U_0)/k_0})$. Since the problem is local on $S_0$ we may assume $S_0 = \mathrm{Spec}\,(R)$. Then one can find a finite ring extension $R \subseteq R' \subseteq k$ such that $R'$ has quotient field $k$ (ex. the integral closure of $R$ in $k$). Again because our problem is local on $S_0$, one may assume $R'/R$ is finite étale. Let $\beta : X_0' := X_0 \times_{\mathrm{Spec}\,(R)} \mathrm{Spec}\,(R') \to X_0$, $u : X_0 \times_{S_0} \kappa(S_0) \to X_0$. Then $u^*\beta_*\beta^*(E, \nabla_E) \cong \alpha_*((E, \nabla_E)|_{X_s/k})$, but $\beta_*\beta^*(E, \nabla_E) \in \mathsf{Mod}_c(D_{X_0/k_0})$. This completes the proof. □

Definition 2.2.4. Let $\mathsf{Mod}_c(D_{S/k}, s)$ be the category whose objects are of the form $(U, M)$, where $U$ is an open subset of $S$ containing $s$ and $M$ is a coherent sheaf on $U$ with a flat connection $\nabla_M$ on $U/k$, whose morphisms between two objects $(U, M)$ and $(U', M')$ are defined by

$$\mathrm{Mor}((U, M), (U', M')) := \mathrm{Hom}_{U \cap U'}((U, M)|_{U \cap U'}, (U', M')|_{U \cap U'}).$$

Let $\mathsf{Mod}_c(D_{X/S/k}, f, s)$ be the category whose objects are of the form $(U, M)$, where $U$ is an open subset of $S$ containing $s$ and $M$ is a coherent sheaf on $f^{-1}(U)$ with a flat connection $\nabla_M$ on $f^{-1}(U)/k$, whose morphisms between two objects $(U, M)$ and $(U', M')$ are defined by

$$\mathrm{Mor}((U,M),(U',M')) := \mathrm{Hom}_{f^{-1}(U\cap U')}((U,M)|_{f^{-1}(U\cap U')},(U',M')|_{f^{-1}(U\cap U')}).$$

PROPOSITION 2.2.5. *Let $X$ be a smooth geometrically connected scheme of finite type over a field $k$ of characteristic 0, $U \subseteq X$ be a dense open subscheme, then for any two objects $(E,\nabla_E),(F,\nabla_F) \in \mathsf{Mod}_c(D_{X/k})$ and any morphism $f_U : (E,\nabla_E)|_{U/S} \to (F,\nabla_F)|_{U/k} \in \mathsf{Mod}_c(D_{U/k})$, we can uniquely extend $f_U$ to a morphism*

$$f : (E,\nabla_E) \to (F,\nabla_F) \in \mathsf{Mod}_c(D_{X/k}).$$

PROOF. The uniqueness is clear since if we have two extensions $f$ and $f'$ then the set of points of $X$ on which $f = f'$ is closed. By the lemma below we may assume $f_U$ is an isomorphism.

Now suppose for any point $x \in X \setminus U$ we can extend $f_U$ to a neighborhood of $x$, then using Zorn's lemma we can extend $f_U$ to a map on $X$. Hence the problem is local. We may assume $X = \mathrm{Spec}\,(A)$ is a smooth integral $k$-algebra with an étale coordinate $X \to \mathbb{A}^r_k$ $(r = \dim X)$, and

$$E = F = A^n := \underbrace{A \oplus \cdots \oplus A}_{n},$$

and $U = \mathrm{Spec}\,(A_f)$ with $f$ non-zero in $A$. Let $d : A^n \to A^n \otimes_A \Omega^1_{A/k}$ be the canonical connection (the $n$-th product of the trivial connections). Adding the $A$-linear map $d - \nabla_E$ on both of the left and the right sides of the following commutative diagram :

$$\begin{tikzcd} A_f^n \arrow[r, "f_U"] \arrow[d, "\nabla_E"'] & A_f^n \arrow[d, "\nabla_F"] \\ A_f^n \otimes_{A_f} \Omega^1_{A_f/k} \arrow[r, "f_U\otimes id"] & A_f^n \otimes_{A_f} \Omega^1_{A_f/k} \end{tikzcd}$$

we get a commutative diagram:

$$\begin{tikzcd} A_f^n \arrow[r, "f_U"] \arrow[d, "d"'] & A_f^n \arrow[d, "d+(\nabla_F-\nabla_E)"] \\ A_f^n \otimes_{A_f} \Omega^1_{A_f/k} \arrow[r, "f_U\otimes id"] & A_f^n \otimes_{A_f} \Omega^1_{A_f/k} \end{tikzcd} .$$

We note that $d+(\nabla_F-\nabla_E)$ is still a flat connection on $A^n$. Let $\{e_i\}_{1\le i\le n}$ be the canonical basis of $A^n$ as a free $A$-module. From the commutative diagram one sees that the image of $e_i$ under $f_U : A_f^n \to A_f^n$ is a horization section of $d+(\nabla_F-\nabla_E)$ on $U$ for each $i$. It suffices to prove the fact that the restriction

$$H^0_{DM}(X,(E,d+(\nabla_F-\nabla_E))) \subseteq H^0_{DM}(U,(E_U,d+(\nabla_F-\nabla_E)))$$

is an isomorphism, since then the image $f_U(e_i)$ is in $A^n$ for each $i$.

Suppose $I = (i_1, i_2, \cdots, i_n)$ be a vector with entries in $\Omega^1_{A/k}$ such that for any vector $v = (v_1, v_2, \cdots, v_n)$ in $A^n$ we have

$$(\nabla_F - \nabla_E)(v) = (v_1 i_1, v_2 i_2, \cdots, v_n i_n).$$

Let $w = (w_1, w_2, \cdots, w_n)$ be a vector in $H^0_{DM}(U, (E_U, d + (\nabla_F - \nabla_E)))$. Then we have

$$(d(w_1), d(w_2), \cdots, d(w_n)) = -(w_1 i_1, w_2 i_2, \cdots, w_n i_n).$$

Now if $w_1 \notin A$, then there is a codimension 1 prime ideal $p \in X$ such that $w_1 = a\pi^{-k}$ with $\pi$ the uniformizer, $a \in A_p$ invertible, and $k > 0$. Then we have

$$d(w_1) = \pi^{-k} d(a) + ka\pi^{-k-1} d(\pi).$$

But we from the second fundamental exact sequence

$$0 \to p/p^2 \to \Omega^1_{A_p/k} \otimes_k A/p \to \Omega^1_{(A/p)/k} \to 0$$

(this sequence is split exact because there is neighborhood of $p$ in which $A/p$ is smooth over $k$) we know that $d(\pi)$ could be extended to a basis of the free $A_p$-module $\Omega^1_{A_p/k}$. This tells us that $\pi^k d(w_1)$ is not a regular differential 1-form in $\Omega^1_{A_p/k}$. This contradicts to the formulae $d(w_1) = -w_1 i_1$ because $i_1$ is a regular 1-form. Thus $w \in A^n$ this is just what we want to show. □

LEMMA 2.2.6. *Let $X$ be a smooth $k$-scheme, $U \subseteq X$ be an open subset, $(E, \nabla_E) \in \mathsf{Mod}_c(D_{X/k})$ and suppose there is an injection*

$$(F', \nabla'_F) \hookrightarrow (E, \nabla_E)|_{U/k} \in \mathsf{Mod}_c(D_{U/k}).$$

*Then there exists a subobject $(F, \nabla_F) \subseteq (E, \nabla_E) \in \mathsf{Mod}_c(D_{X/k})$ such that $(F, \nabla_F)|_{U/k} = (F', \nabla'_F)$ as subojects of $(E, \nabla_E)|_{U/k}$*

PROOF. Let $j : U \subseteq X$ be the inclusion. We take $F$ to be the inverse image of $j_*F'$ under the adjunction map $E \to j_*j^*E$. Then $F$ is a coherent sheaf, and one checks easily that $F \to E \xrightarrow{\nabla_E} E \times_{O_X} \Omega^1_{X/k}$ factors through $F \times_{O_X} \Omega^1_{X/k} \to E \times_{O_X} \Omega^1_{X/k}$. Hence $F$ is equipped with a connection $\nabla_F$ and becomes a subobject of $(E, \nabla_E)$. Clearly $(F, \nabla_F)|_{U/k}$ is equal to $(F', \nabla_{F'})$ as subojects (since $F|_U$ is equal to $F'$). □

The above Proposition and the above Lemma implies immediately the following:

LEMMA 2.2.7. *The category $\mathsf{Mod}_c(D_{S/k}, S)$ (resp. $\mathsf{Mod}_c(D_{X/S/k}, f, s)$) is an abelian $k$-linear rigid tensor tensor category equipped with an exact faithful $k$-linear tensor functor $(M, U) \mapsto M|_s$ (resp. $(M, U) \mapsto M|_x$). Thus it is a neutral Tannakian category, so we have a Tannakian group $\hat{\pi}^{alg}(S, s)$ (resp.$\hat{\pi}^{alg}(X, x)$) associated to $\mathsf{Mod}_c(D_{S/k}, s)$ (resp. $\mathsf{Mod}_c(D_{X/S/k}, f, s)$). Furthermore, the canonical functor $\mathsf{Mod}_c(D_{S/k}) \to \mathsf{Mod}_c(D_{S/k}, s)$ (resp. $\mathsf{Mod}_c(D_{X/k}) \to \mathsf{Mod}_c(D_{X/S/k}, f, s)$) is fully faithful and stable under taking sub quotients. Thus we get a canonical surjection $\hat{\pi}^{alg}(S, s) \twoheadrightarrow \pi^{alg}(S, s)$ (resp. $\hat{\pi}^{alg}(X, x) \twoheadrightarrow \pi^{alg}(X, x)$).*

Using the results in the previous sections and apply our above lemma to $\mathsf{Mod}_c(D_{X/S/k}, f, s)$ and $\mathsf{Mod}_c(D_{S/k}, s)$ we get:

THEOREM 2.2.8. *The homotopy sequence*

$$\pi^{alg}(X_s, x) \to \hat{\pi}^{alg}(X, x) \to \hat{\pi}^{alg}(S, s) \to 1$$

*is exact. And one has a commutative diagram:*

$$\begin{tikzcd} \pi^{alg}(X_s,x) \arrow[r] \arrow[d, equal] & \hat{\pi}^{alg}(X,x) \arrow[r] \arrow[d, two heads] & \hat{\pi}^{alg}(S,s) \arrow[r] \arrow[d, two heads] & 1 \\ \pi^{alg}(X_s,x) \arrow[r] & \pi^{alg}(X,x) \arrow[r] & \pi^{alg}(S,s) \arrow[r] & 1 \end{tikzcd}$$

Theorem 2.2.9. *Under the hypothesis in the beginning of this subsection, the homotopy sequence*

$$\pi^{alg}(X_s,x) \to \pi^{alg}(X,x) \to \pi^{alg}(S,s) \to 1$$

*is exact.*

Proof. From the surjectivity of $\hat{\pi}^{\mathrm{alg}}(X,x) \twoheadrightarrow \pi^{\mathrm{alg}}(X,x)$ we know that the image of $\pi^{\mathrm{alg}}(X_s,x)$ is a normal subgroup of $\pi^{\mathrm{alg}}(X,x)$. Then we take $G := \mathrm{Coker}(\pi^{\mathrm{alg}}(X_s,x) \to \pi^{\mathrm{alg}}(X,x))$, and we get a surjective map of group schemes $G \twoheadrightarrow \pi^{\mathrm{alg}}(S,s)$. From condition (a) we know the functor

$$\mathsf{Rep}_k(\pi^{\mathrm{alg}}(S,s)) \to \mathsf{Rep}_k(G)$$

is essentially surjective, while the surjectivity of $G \twoheadrightarrow \pi^{\mathrm{alg}}(S,s)$ tells us that

$$\mathsf{Rep}_k(\pi^{\mathrm{alg}}(S,s)) \to \mathsf{Rep}_k(G)$$

is an equivalence of categories. This finishes the proof. □

**2.3. The general case.** In this subsection we come to the general case: $f : X \to S$ be a proper smooth morphism between two smooth connected schemes of finite type over a field $k$ of characteristic 0 with geometrically connected fibres, $x \in X(k)$, $s \in S(k)$ and $f(x) = s$.

Proposition 2.3.1. *If $k \subseteq k'$ is a field extension, $f', X', S', x', s'$ are the corresponding morphism, schemes, points obtained by base change, and if the sequence*

$$\pi^{alg}(X'_{s'},x') \to \pi^{alg}(X',x') \to \pi^{alg}(S',s') \to 1$$

*is exact as $k'$-group schemes, then the sequence of $k$-group schemes*

$$\pi^{alg}(X_s,x) \to \pi^{alg}(X,x) \to \pi^{alg}(S,s) \to 1$$

*is also exact.*

Proof. Let $\mathfrak{C}(X')$ be the full subcategory of $\mathsf{Mod}_c(D_{X'/k'})$ whose objects, after being pushed forward along the projection $X' \to X$, are the inductive limits of their coherent subojects (i.e. subojects belong to $\mathsf{Mod}_c(D_{X/k})$). This $\mathfrak{C}(X')$ is a Tannakian subcategory and its Tannakian group is precisely $\pi^{\mathrm{alg}}(X,x) \times_k k'$ [**De1**][10.38, 10.41]. But it is clear that this full subcategory is also stable under taking subquotients. Thus the canonical map $\pi^{\mathrm{alg}}(X',x') \to \pi^{\mathrm{alg}}(X,x) \times_k k'$ is surjective. The same argument applies to $X_s$ and $S$.

Hence we get a commutative diagram with the first row being exact

$$\begin{tikzcd} \pi^{\mathrm{alg}}(X'_{s'}, x') \arrow[r, "a'"] \arrow[d, two heads] & \pi^{\mathrm{alg}}(X', x') \arrow[r] \arrow[d, two heads] & \pi^{\mathrm{alg}}(S', s') \arrow[r] \arrow[d, two heads] & 1 . \\ \pi^{\mathrm{alg}}(X_s, x) \times_k k' \arrow[r, "a"] & \pi^{\mathrm{alg}}(X, x) \times_k k' \arrow[r] & \pi^{\mathrm{alg}}(S, s) \times_k k' \arrow[r] & 1 \end{tikzcd}$$

Since the image of $a'$ is normal, so the image of $a$ is also normal. Hence the image of $\pi^{\mathrm{alg}}(X_s, x) \to \pi^{\mathrm{alg}}(X, x)$ is normal. Using the same argument employed in Theorem 2.2.9 we conclude the proof of this proposition. $\square$

THEOREM 2.3.2. *Let $f : X \to S$ be a proper smooth morphism between two smooth connected schemes of finite type over a field $k$ of characteristic 0 with geometrically connected fibres, $x \in X(k)$, $s \in S(k)$ and $f(x) = s$. Then the homotopy sequence*

$$\pi^{alg}(X_s, x) \to \pi^{alg}(X, x) \to \pi^{alg}(S, s) \to 1$$

*is exact.*

PROOF. Since condition (a) (b) and surjectivity have been proved in §2.1, so we only need to check condition (c). But for any object $(E, \nabla_E) \in \mathsf{Mod}_c(D_{X/k})$ and any morphism

$$\delta : (F', \nabla_{F'}) \subseteq (E, \nabla_E)|_{X_s/k} \in \mathsf{Mod}_c(D_{X_s/k}),$$

there is a finitely generated field over $\mathbb{Q}$ on which all these objects $(X,\ S,\ (E, \nabla_E),\ \cdots)$ and morphisms $(f,\ x,\ s,\ \delta,\ \cdots)$ are defined. So we can reduce our problem to the case when $k$ is a finitely generated field over $\mathbb{Q}$. But in light of the previous proposition we can assume our field $k$ is actually $\mathbb{C}$.

Let $K$ be the algebraic closure of the function field of $S$. Since $K$ and $\mathbb{C}$ have the same transcendental degree over $\mathbb{Q}$, they are isomorphic as fields. Now $\eta : \mathrm{Spec}\,(K) \hookrightarrow S$ is a geometric generic point, so by the discussion in §2.2 the sequence

$$\pi^{\mathrm{alg}}(X_\eta, \eta') \to \pi^{\mathrm{alg}}(X_K, \eta') \to \pi^{\mathrm{alg}}(S_K, \eta) \to 1$$

is exact (where $X_K$ (resp. $S_K$) is the base change of $X$ (resp. $S$) from $k$ to $K$, and $\eta'$ is any chosen $K$-rational point of $X_K$ above $\eta$). From the lemma below we get a commutative diagram of $K$-group schemes

$$\begin{tikzcd} \pi^{\mathrm{alg}}(X_\eta, \eta') \arrow[r] \arrow[d, "\cong"] & \pi^{\mathrm{alg}}(X_K, \eta') \arrow[r] \arrow[d, "\cong"] & \pi^{\mathrm{alg}}(S_K, \eta) \arrow[r] \arrow[d, "\cong"] & 1 . \\ \pi^{\mathrm{alg}}(X_{s_K}, x_K) \arrow[r] & \pi^{\mathrm{alg}}(X_K, x_K) \arrow[r] & \pi^{\mathrm{alg}}(S_K, s_K) \arrow[r] & 1 \end{tikzcd}$$

Thus the last row is exact. Then we can conclude our theorem by our previous proposition. $\square$

LEMMA 2.3.3. *If $f : X \to S$ is a smooth proper morphism between two smooth quasi-compact geometrically connected $\mathbb{C}$-schemes with geometrically connected fibres, $x, x'$ and $s, s'$ are $\mathbb{C}$-rational points of $X$ and $S$ respectively with $f(x) = s$ and $f(x') = s'$, then there* *exists a commutative diagramme of $\mathbb{C}$-group schemes*

$$\begin{tikzcd}
\pi^{alg}(X_{s'}, x') \arrow[r] \arrow[d, "\cong"] & \pi^{alg}(X, x') \arrow[r] \arrow[d, "\cong"] & \pi^{alg}(S, s') \arrow[d, "\cong"] \\
\pi^{alg}(X_s, x) \arrow[r] & \pi^{alg}(X, x) \arrow[r] & \pi^{alg}(S, s)
\end{tikzcd}.$$

Proof. From the sequences of $\mathbb{C}$-schemes:

$$X_s \to X \xrightarrow{f} S \qquad \text{and} \qquad X_{s'} \to X \xrightarrow{f} S$$

one gets sequences of analytic spaces:

$$X_s^{an} \to X^{an} \xrightarrow{f^{an}} S^{an} \qquad \text{and} \qquad X_{s'}^{an} \to X^{an} \xrightarrow{f^{an}} S^{an},$$

where $X_s^{an}$ (resp. $X_{s'}^{an}$) is still the fibre of $s \in S^{an}$ (resp. $s' \in S^{an}$) under $f^{an}$, since the functor $-^{an}$ commutates with fibre product [**SGA1**][Exposé XII, 1.2]. Now applying the first homotopy functor (in topology) to these analytic spaces one gets a commutative diagram:

$$\begin{tikzcd}
\pi_1^{\text{top}}(X_{s'}^{an}, x') \arrow[r] & \pi_1^{\text{top}}(X^{an}, x') \arrow[r] \arrow[d, "\cong"] & \pi_1^{\text{top}}(S^{an}, s') \arrow[d, "\cong"] \\
\pi_1^{\text{top}}(X_s^{an}, x) \arrow[r] & \pi_1^{\text{top}}(X^{an}, x) \arrow[r] & \pi_1^{\text{top}}(S^{an}, s)
\end{tikzcd}.$$

In fact by carefully choosing a path between $x$ and $x'$, there exists a group isomorphism

$$\pi_1^{\text{top}}(X_s^{an}, x) \xrightarrow{\cong} \pi_1^{\text{top}}(X_{s'}^{an}, x')$$

making the above diagramme commutative.

To show this one first defines a subset $Z \subseteq S^{an}$ consists of points $t \in S^{an}$ which admits a point $y \in X^{an}$, a path $\alpha$ between $x$ and $y$, and an isomorphism

$$\pi_1^{\text{top}}(X_s^{an}, x) \xrightarrow{\cong} \pi_1^{\text{top}}(X_t^{an}, y)$$

making the diagram

$$\begin{tikzcd}
\pi_1^{\text{top}}(X_s^{an}, x) \arrow[r] \arrow[d, "\cong"] & \pi_1^{\text{top}}(X^{an}, x) \arrow[d, "\alpha"] \\
\pi_1^{\text{top}}(X_t^{an}, y) \arrow[r] & \pi_1^{\text{top}}(X^{an}, y)
\end{tikzcd}$$

commutative. $Z$ is both open and closed, since for any $t \in S^{an}$ by Ehresmann's theorem ($f^{an}$ is proper smooth by [**SGA1**][Exposé XII, proposition 3.1 et proposition 3.2]) one knows that in a neighborhood $U$ of $t \in S^{an}$ $f^{an-1}(U)$ is isomorphic to $X_t^{an} \times S^{an}$ as a topological space. Thus for any $t' \in U$ one gets a commutative diagram

$$\begin{tikzcd}
\pi_1^{\mathrm{top}}(X_t^{an}, y) \arrow[r] \arrow[d, "\cong"] & \pi_1^{\mathrm{top}}(X^{an}, y) \arrow[d, "\cong"] \\
\pi_1^{\mathrm{top}}(X_{t'}^{an}, y') \arrow[r] & \pi_1^{\mathrm{top}}(X^{an}, y')
\end{tikzcd}$$

by choosing any points $y, y' \in f^{an-1}(U)$ and any path between them (inside $f^{an-1}(U)$). Hence $t \in Z$ if and only if $t' \subseteq Z$. This shows that $Z$ is both open and closed. On the other hand, we know that $S^{an}$ is connected ([**SGA1**][Exposé XII, proposition 2.4]). Thus $Z = S^{an}$ and hence $s' \in Z$.

Now let us denote the category of integrable analytic connections on $X^{an}$ and $X_s^{an}$, $X_{s'}^{an}$ by $\mathrm{Conn}(X^{an})$ and $\mathrm{Conn}(X_s^{an})$, $\mathrm{Conn}(X_{s'}^{an})$ respectively. By Riemann-Hilbert correspondence one has a 2-commutative diagram of neutral Tannakian categories (i.e. the $k$-linear tensor functors in the diagram also respect the fibres functors):

$$\begin{tikzcd}
 & \mathrm{Conn}(X_s^{an}) \arrow[r, "\cong"] \arrow[dd, "\cong"] & \mathsf{Rep}_{\mathbb{C}}(\pi_1^{\mathrm{top}}(X_s^{an}, x)) \arrow[dd, "\cong"] \\
\mathrm{Conn}(X^{an}) \arrow[ur, "\lambda"] \arrow[dr, "\lambda'"'] & & \\
 & \mathrm{Conn}(X_{s'}^{an}) \arrow[r, "\cong"] & \mathsf{Rep}_{\mathbb{C}}(\pi_1^{\mathrm{top}}(X_{s'}^{an}, x'))
\end{tikzcd}$$

Let us set $\iota$ to be the canonical functor $\mathsf{Mod}_c(D_{X/\mathbb{C}}) \to \mathrm{Conn}(X^{an})$ sending an integrable algebraic connection on $X$ to an integrable analytic connection on $X^{an}$. This functor gives a 2-commutative diagram of neutral Tannakian categories:

$$\begin{tikzcd}
 & & \mathsf{Mod}_c(D_{X_s/\mathbb{C}}) \arrow[r, "\cong"] & \mathrm{Conn}(X_s^{an}) \arrow[dd, "\cong"] \\
\mathsf{Mod}_c(D_{S/\mathbb{C}}) \arrow[r] & \mathsf{Mod}_c(D_{X/\mathbb{C}}) \arrow[r, "\iota"] \arrow[ur, "\widetilde{\lambda}"] \arrow[dr, "\widetilde{\lambda}'"'] & \mathrm{Conn}(X^{an}) \arrow[ur, "\lambda"] \arrow[dr, "\lambda'"'] & \\
 & & \mathsf{Mod}_c(D_{X_{s'}/\mathbb{C}}) \arrow[r, "\cong"] & \mathrm{Conn}(X_{s'}^{an})
\end{tikzcd}$$

Now applying Tannakian duality we conclude the proof of our lemma. $\square$

## 3. The Künneth formula in characteristic $p > 0$

There are good reasons to expect the homotopy sequence to be exact for algebraic fundamental groups in characteristic $p$. For one thing the homotopy sequence is exact in characteristic 0, for another the étale fundamental group which can be considered as the profinite completion of the algebraic fundamental group has exact homotopy sequence. But the proof of the exactness is already very complicated in characteristic 0, and even worse we used some transcendental methods in the end of that proof. So up to now we still can not prove that. However we can prove that in a special case–the Künneth formula. This provides some further evidence that the homotopy sequence should be exact in characteristic $p$.

We have showed in Chapter 1, §3, that $\mathrm{Strat}(X/k)$ is an abelian $k$-linear rigid tensor category and we have a $k$-linear tensor equivalence between $\mathrm{Strat}(X/k)$ and $\mathsf{Mod}_c(D_{X/k})$. If $x \in X(k)$ is a rational point then we get two natural fibre functors for $\mathrm{Strat}(X/k)$ and $\mathsf{Mod}_c(D_{X/k})$ which are compatible with the above equivalence. Thus $\pi^{\mathrm{alg}}(X, x)$ can be equally defined via $\mathrm{Strat}(X/k)$. So we will only work with $\mathrm{Strat}(X/k)$ in the rest of this section.

**3.1. Notations and Conventions.** Now we fix a smooth map $f : X \to S$ between two smooth geometrically connected schemes over a field $k$ of characteristic $p > 0$. Let $(E^{(i)}, \sigma^{(i)})$ be a stratified bundle over $X/k$. We denote by $(E^{(i)}_S, \sigma^{(i)}_S)$ the pull-back of $(E^{(i)}, \sigma^{(i)})$ along the canonical map $\lambda_i : X^{(i)}_S \to X^{(i)}$.

We first observe that all the relative Frobenius and absolute Frobenius are homoemorphismes on the ambient spaces. Thus if we have sheaves of abelian groups on $X^{(i)}$ and $X^{(i)}_S$, then we can actually regard them as sheaves of abelian groups on $X^{(0)} = X$. We have a canonical imbedding of sheave of abelian groups $E^{(i)} \hookrightarrow E^{(i)} \otimes_{O_{X^{(i)}}} O_X \cong E^{(0)}$ for each $i$. Thus each $E^{(i)}$ can be seen as a subsheaf of abelian groups of $E^{(0)}$. Similarly each $E^{(i)}_S$ is also a sub sheaf of abelian groups of $E^{(0)}_S = E^{(0)}$. Let us write $E^{(\infty)}_S$ for the intersection of all $E^{(i)}_S$ as sub sheaves of abelian groups of $E^{(0)}$. Then for each $i \in \mathbb{N}$ the sheaf $F^{(i)} := f^{(i)}_* E^{(\infty)}_{S^{(i)}}$ of abelian groups is naturally equipped with a structure of an $O_{S^{(i)}}$-module, where $f^{(i)} : X^{(i)} \to S^{(i)}$ is the canonical structure map.

For each $i \in \mathbb{N}$, we write $\phi_i : S \to S^{(i)}$ for the $i$-th relative Frobenius of $S$ and $\varphi_i : X \to X^{(i)}$ for the $i$-th relative Frobenius of $X$. The notations are indicated in the following diagram:

$$\begin{tikzcd}
X \arrow[dr, dotted] \arrow[drr, bend left, "\varphi_i"] \arrow[ddr, bend right, "f"'] & & \\
& X^{(i)}_S \arrow[r, "\lambda_i"] \arrow[d, "f^{(i)}_S"] & X^{(i)} \arrow[d, "f^{(i)}"] \\
& S \arrow[r, "\phi_i"] & S^{(i)}
\end{tikzcd}.$$

**3.2. Some Preparations.** The following discussion is contained in [**Hai**]. We will restate the results and give some variant of his proof for the convenience of the reader.

Theorem 3.2.1. (Phùng Hô Hai) *Notations being as in §3.1, we have a canonical $O_{S^{(i)}}$-module isomorphism $\tau^{(i)} : \phi_i^* F^{(i+1)} \cong F^{(i)}$ for all $i \in \mathbb{N}$.*

Proof. One can easily set up an induction argument to reduce our proof to the case when $i = 0$. Now for each $i \in \mathbb{N}$, the above diagram gives us a canonical isomorphism

$$\xi_i : \phi_i^* f^{(i)}_* E^{(i)} \xrightarrow{\cong} (f^{(i)}_S)_* E^{(i)}_S.$$

One also has canonical imbeddings (as sheaves of $O_S$-modules)

$$\phi_{i+1}^* f_*^{(i+1)} E^{(i+1)} \subseteq \phi_i^* f_*^{(i)} E^{(i)} \quad \text{and} \quad (f_S^{(i+1)})_* E_S^{(i+1)} \subseteq (f_S^{(i)})_* E_S^{(i)}.$$

And these imbeddings are compatible with these $\xi_i$ in the obvious way. So we get a filtration of $\phi_1^* f_*^{(1)} E^{(1)}$ and a filtration of $(f_S^{(1)})_* E_S^{(1)}$ such that $\xi_1$ preserves the filtration. But the intersection of the filtration of the left side

$$\bigcap_{i=1}^{\infty} \phi_i^* f_*^{(i)} E^{(i)}$$

is precisely $\phi_1^* f_*^{(1)} E_{S^{(1)}}^{(\infty)}$, while the intersection on the other side gives us $f_* E_S^{(\infty)}$. Thus $\xi_1$ induces the isomorphism $\tau^{(0)}$. □

Remark. If $f : X \to S$ is in addition proper, we can define a functor $H^0_{\mathrm{str}}(X/S, -)$ from the category of stratified bundles over $X/k$ to the category of stratified bundles over $S/k$.

Theorem 3.2.2. (Phùng Hô Hai) *Notations being as above. If $f : X \to S$ is in addition proper, then for any point $i : s \hookrightarrow S$, $H^0_{\mathrm{str}}(X/S, -)$ satisfies base change for the following diagram:*

$$\begin{tikzcd} X_s \arrow[r, "i'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ s \arrow[r, "i"] & S \end{tikzcd},$$

*i.e. for any stratified bundle $(E^{(i)}, \sigma_i)$ on $X/k$ the canonical map*

$$i^* H^0_{\mathrm{str}}(X/S, (E^{(i)}, \sigma_i)) \to H^0_{\mathrm{str}}(X_s/s, i'^*(E^{(i)}, \sigma_i))$$

*is an isomorphism.*

Proof. For a proof see [**Hai**][Corollary 2.9]. There the field is assumed to be algebraically closed and the points are rational points, but the same proof holds for not necessarily algebraically closed field and non-rational points. □

Now we assume $f : X \to S$ is also (in addition to the assumptions in §3.1) proper surjective and has geometrically connected fibres. Then we have the following:

Proposition 3.2.3. (Phùng Hô Hai) *The adjunction map*

$$f^* H^0_{\mathrm{str}}(X/S, (E^{(i)}, \sigma^{(i)})) \to (E^{(i)}, \sigma^{(i)})$$

*is injective in* $\mathrm{Strat}(X/k)$*. If $S = \mathrm{Spec}\,(k)$, then the adjunction map gives the maximal trivial subobject of $(E^{(i)}, \sigma^{(i)})$ in* $\mathrm{Strat}(X/k)$*.*

Proof. Let $K$ be the kernel of the adjunction map. So we get an exact sequence

$$0 \to K \to f^* H^0_{\mathrm{str}}(X/S, (E^{(i)}, \sigma^{(i)})) \to (E^{(i)}, \sigma^{(i)}).$$

But since $K$ is a subobject of $f^* H^0_{\mathrm{str}}(X/S, (E^{(i)}, \sigma^{(i)}))$, so for any $s \in S$, $K$ after restricting to $X_s/\kappa(s)$ is a suboject of a trivial suboject, so the restriction of $K$ is trivial. Because $f$ is proper separable and $O_S \cong f_*O_X$, there is an object $K' \in \mathrm{Strat}(S/k)$ such that $f^*K' = K$. Now we apply the left exact functor $H^0_{\mathrm{str}}(X/S, -)$ to the sequence

$$0 \to f^*K' \to f^*H^0_{\mathrm{str}}(X/S, (E^{(i)}, \sigma^{(i)})) \to (E^{(i)}, \sigma^{(i)}).$$

Thus we get an exact sequence in $\mathrm{Strat}(S/k)$:

$$0 \to K' \to H^0_{\mathrm{str}}(X/S, (E^{(i)}, \sigma^{(i)})) \xrightarrow{=} H^0_{\mathrm{str}}(X/S, (E^{(i)}, \sigma^{(i)})).$$

This tells us that $K' = 0$, so the adjunction map is injective.

If $S = \mathrm{Spec}\,(k)$ then $f^*H^0_{\mathrm{str}}(X/k, (E^{(i)}, \sigma^{(i)})) \subseteq (E^{(i)}, \sigma^{(i)})$ is a trivial subobject. If there is another trivial suboject $(F^{(i)}, \tau^{(i)}) \subseteq (E^{(i)}, \sigma^{(i)})$ then we apply the functor $f^*H^0_{\mathrm{str}}(X/k, -)$ to the imbedding. So we get

$$\begin{tikzcd}
f^*H^0_{\mathrm{str}}(X/k, (F^{(i)}, \tau^{(i)})) \arrow[r] \arrow[d, "\cong"] & f^*H^0_{\mathrm{str}}(X/k, (E^{(i)}, \sigma^{(i)})) \arrow[d] \\
(F^{(i)}, \tau^{(i)}) \arrow[r] & (E^{(i)}, \sigma^{(i)})
\end{tikzcd}.$$

The left vertical arrow is an isomorphism because $(F^{(i)}, \tau^{(i)})$ is trivial. This diagram concludes our proof of the second statement. $\square$

### 3.3. The Künneth formula.

THEOREM 3.3.1. *If $X$ and $Y$ are smooth geometrically connected schemes over a field $k$ of characteristic $p > 0$, and if $Y$ is proper over $k$, $x \in X(k)$, $y \in Y(k)$. Then the canonical map*

$$\pi^{alg}(X \times_k Y, (x, y)) \to \pi^{alg}(X, x) \times_k \pi^{alg}(Y, y)$$

*is an isomorphism.*

PROOF. One has the following commutative diagram induced where all the maps are canonical:

$$\begin{tikzcd}
1 \arrow[r] & \pi_1^{\mathrm{alg}}(Y, y) \arrow[r] \arrow[d, equal] & \pi_1^{\mathrm{alg}}(X \times_k Y, (x, y)) \arrow[r] \arrow[d] & \pi_1^{\mathrm{alg}}(X, x) \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \pi_1^{\mathrm{alg}}(Y, y) \arrow[r] & \pi_1^{\mathrm{alg}}(X, x) \times_k \pi_1^{\mathrm{alg}}(Y, y) \arrow[r] & \pi_1^{\mathrm{alg}}(X, x) \arrow[r] & 1
\end{tikzcd}.$$

So to show that the middle map is an isomorphism it is enough to show that the first row is exact. Thus we only need to show the homotopy sequence is exact for the projection $X \times_k Y \to X$.

By Theorem 4.12 and Proposition 4.13 we have condition (a) and (b). Now we check the condition (c). Let us write out the maps as follows

$$X \xrightarrow{id \times y} X \times_k Y \xrightarrow{pr_1} X.$$

Condition (c) says that for any $(E^{(i)}, \sigma^{(i)})) \in \mathrm{Strat}(X \times_k Y)$ and any imbedding

$$(F_1^{(i)}, \tau_1^{(i)}) \subseteq (id \times y)^*(E^{(i)}, \sigma^{(i)})) \in \mathrm{Strat}(X/k)$$

there is an object $(F^{(i)}, \tau^{(i)}) \in \mathrm{Strat}(X \times_k Y/k)$ and a surjection

$$(id \times y)^*(F^{(i)}, \tau^{(i)}) \twoheadrightarrow (F_1^{(i)}, \tau_1^{(i)}).$$

In our case we can take $(F^{(i)}, \tau^{(i)})$ to be $pr_1^*(F_1^{(i)}, \tau_1^{(i)})$ then we have $(id \times y)^*(F^{(i)}, \tau^{(i)}) \cong (F_1^{(i)}, \tau_1^{(i)})$. This proves (c). $\square$

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