The Nori Local Fundamental Groups

June 11, 2022

Lei ZHANG

The Chinese University of Hong Kong

The Definition of the Nori Fundamental Group

Let X be a connected reduced scheme over a field k, and let $x \in X$ be a point of X. Then we can consider the category $N(X/k, x)$ of triples (P, G, p), where G is a finite k-group scheme, P is an fppf G-torsor over X and p is a point of P lying above x. It is proven by Nori that the category $N(X/k, x)$ is cofiltered.

Definition

The *Nori fundamental group* of X at the point x is defined to be

$$\pi_1^{\mathrm{N}}(X, x) := \varprojlim_{(P_i, G_i, p_i) \in N(X/k, x)} G_i$$

which is a profinite group scheme over k.

Question

How should we understand $\pi_1^{\mathrm{N}}(X, x)$?

Remark

If x is taken to be a geometric point, then $N(X/k, x)$ is just the category of pointed finite torsors over X. If we take $N_c(X/k, x) \subseteq N(X/k, x)$ to be the full subcategory of triples (P, G, p) where G is finite and constant, then $N_c(X/k, x)$ is *basically* the category of Galois covers of X. Thus

$$\pi_1^{\text{ét}}(X, x) = \varprojlim_{(P_i, G_i, p_i) \in N_c(X/k, x)} G_i$$

and there is an obvious surjection:

$$\pi_1^{\mathsf{N}}(X, x) \twoheadrightarrow \pi_1^{\text{ét}}(X, x)$$

So π_1^{N} is so to speak a "generalization" of the étale fundamental group, especially when k is an algebraically closed field of characteristic 0, then $\pi_1^{\mathsf{N}} = \pi_1^{\text{ét}}$.

Question

What are the “new elements” in the Nori fundamental group compared with the étale fundamental group?

Question

What are the "new elements" in the Nori fundamental group compared with the étale fundamental group?

Example

Let k be a field of characteristic $p > 0$. The Kummer cover

$$\mathbb{G}_{m,k} \xrightarrow{x \mapsto x^p} \mathbb{G}_{m,k}$$

is a $\mu_{p,k}$-torsor, where $\mu_{p,k}$ the "group of p-th roots of unity", which is a finite *local group scheme* over k.

Question

How to “compute” the Nori fundamental group?

Theorem (Nori)

In particular, if X is a geometrically connected and geometrically reduced ***proper*** *scheme over k, and if $x \in X$ is a k-rational point, then we have a natural equivalence*

$$\mathrm{Rep}_k(\pi^{\mathrm{N}}(X,x)) \xrightarrow{\simeq} \mathrm{EFin}(X).$$

Recall a vector bundle V on X is called *finite* if there exists $f(T), g(T) \in \mathbb{N}[T]$ such that $f(V) \cong g(V)$. The vector bundle V is called *essentially finite* if it is the kernel of a morphism of two finite vector bundles.

Note that an essentially finite vector bundle is semistable of slope 0. If k is of characteristic 0, then an essentially finite vector bundle is finite.

Corollary

If $X = \mathbb{P}^n_k$ *and* $x \in X$ *is a k-rational point, then* $\pi_1^{\mathrm{N}}(X, x)$ *is trivial.*

Remark

Indeed more is true: All complete normal rational varieties have a trivial Nori fundamental group. There is one non-trivial computation though - the computation of the Nori fundamental group of an abelian variety which is due to Nori. For an abelian variety X over an algebraically closed field k we have

$$\pi^{\mathrm{N}}(X, x) = \varprojlim_{n \in \mathbb{N}} X_n$$

where X_n is n-torsion part of X.

Question

The Nori fundamental group is hard to compute, but can we make our lives easier while still keeping the "essentially new elements" of Nori?

Question

The Nori fundamental group is hard to compute, but can we make our lives easier while still keeping the "essentially new elements" of Nori?

Answer

Yes, we can look at the *local quotient* of the Nori fundamental group.

Recall a finite group scheme G over a field k is called *local* or *infinitesimal* if it is connected.

Definition

Let X be a reduced scheme over a field k. Let $N_l(X/k)$ be the category of pairs (P, G), where G is a finite local k-group scheme and P is a G-torsor over X. Then $N_l(X/k)$ is cofiltered. Thus we have a profinite k-group scheme

$$\pi_1^{\mathrm{L}}(X/k) := \varprojlim_{(P_i, G_i) \in N_l(X/k)} G_i$$

Just as $\pi_1^{\text{ét}}(X, x_0)$, the fundamental group $\pi_1^{\mathrm{L}}(X/k)$ is also a quotient of $\pi_1^{\mathrm{N}}(X, x_0)$ - the maximal pro-local quotient.

Theorem (Tonini-Z)

Let X be a scheme geometrically reduced over a field k of characteristic $p > 0$. Let $\mathcal{D}_i$ be the category of triples $(\mathcal{F}, V, \lambda)$, where $\mathcal{F} \in \mathrm{Vect}(X)$, $V \in Vect_k$ and $\lambda\colon F_X^{i*}\mathcal{F} \cong V \otimes_k \mathcal{O}_X$. Then $\mathcal{D}_i$ is a k-Tannakian category with the k-structure given by

$$k \longrightarrow \mathrm{End}(\mathcal{O}_X, k, \mathrm{id}); \quad x \mapsto (x, x^{p^i})$$

We have fully faithful k-linear exact monoidal transition functors

$$\mathcal{D}_i \longrightarrow \mathcal{D}_{i+1}; \quad (\mathcal{F}, V, \lambda) \mapsto (\mathcal{F}, F_k^* V, F_X^* \lambda)$$

Passing to the limit we have $\mathcal{D}_\infty := \varinjlim_{i \in \mathbb{N}} \mathcal{D}_i$. Then there is a natural equivalence

$$\mathrm{Rep}_k(\pi_1^{\mathrm{L}}(X/k)) \xrightarrow{\cong} \mathcal{D}_\infty$$

when k is perfect.

Theorem (Romagny-Tonini-Z)

Let K be a field over a perfect field k. Let $\mathsf{PI}(K)$ denote the category of purely inseparable field extensions of K, then the natural functor

$$\mathsf{PI}(K) \longrightarrow \{\textit{Affine } k\textit{-group schemes over } \pi_1^{\mathsf{L}}(K/k)\}$$

$$K'/K \longmapsto (\pi_1^{\mathsf{L}}(\mathrm{Spec}(K')/k) \to \pi_1^{\mathsf{L}}(\mathrm{Spec}(K)/k))$$

is fully faithful.

We first show that for $K'/K \in \mathsf{PI}(K)$ the map of fundamental groups $\pi_1^{\mathsf{L}}(\mathrm{Spec}(K')/k) \to \pi_1^{\mathsf{L}}(\mathrm{Spec}(K)/k)$ is injective. Thus the functor is actually a functor

$$\mathsf{PI}(K) \longrightarrow (\text{closed subgroups of } \pi_1^{\mathsf{L}}(\mathrm{Spec}(K)/k))$$

Remark

- The category $PI(K)$ is very simple: Any two objects $K_1, K_2 \in PI(K)$ has a morphism iff one field is "contained" in the other. Note that since these purely inseparable extensions, there is a unique way for a field to be contained in the other.
- We prove that if $K' \in PI(K)$, then the induced map $\pi_1^{\mathsf{L}}(K'/k) \to \pi_1^{\mathsf{L}}(K/k)$ is injective. Thus the functor above provides a order preserving embedding of ordered sets.
- If one considers the separable extensions of the field K, then the functor would be much more difficult: One needs a choice of an algebraic closure (the base point) to produce a fundamental group, and different base points lead to non-canonically isomorphic fundamental groups. This poses an essential difficulty in understanding the functor.

Question

Do we have another way of “compute” the fundamental groups?

Question

Do we have another way of "compute" the fundamental groups?

Answer

For a profinite group (scheme) π, its set of finite quotients

$$\pi_A := \{\text{The set of finite quotients of } \pi\}$$

encodes essential information about π. Thus one can "compute" a profinite group π by studying its π_A.

Let X be a smooth connected projective curve over an algebraically closed field k of characteristic $p > 0$. Let U be an affine curve obtained from X by deleting n points. Let g be the genus of X.

Theorem (M. Raynaud, D. Harbater)

A finite group $G \in \pi_A^{\text{ét}}(U) = \{$finite quotients of $\pi_1^{\text{ét}}(U)\}$ if and only if there is a continuous surjection:

$$\hat{\Gamma}_{g,n}^{(p')} \twoheadrightarrow G^{(p')}$$

where $\Gamma_{g,n}$ is the free group of rank $2g + n - 1$, $\hat{\Gamma}_{g,n}$ is the profinite completion and $(-)^{(p')}$ is the maximal prime to p quotient.

Conjecture (S. Otabe)

Let $\gamma := \dim_{\mathbb{F}_p} \mathrm{Pic}^0_X[p](k)$*. Is it true that a finite local group* k*-group scheme* $G \in \pi^{\mathrm{L}}_A(U) = \{$*finite quotients of* $\pi^{\mathrm{L}}_1(U)\}$ *if and only if there is an injection:*

$$\mathrm{Hom}_{\mathrm{grp.sch}}(G, \mathbb{G}_{m,k}) \subseteq (\mathbb{Q}_p/\mathbb{Z}_p)^{\gamma+n-1}?$$

S. Otabe showed that

$$\begin{aligned}(\mathbb{Q}_p/\mathbb{Z}_p)^{\gamma+n-1} &= \mathrm{Hom}_{\mathrm{grp.sch}}(\pi^{\mathrm{L}}(U), \mathbb{G}_{m,k}) \\ &= \mathrm{Hom}_{\mathrm{grp.sch}}(\pi^{\mathrm{L}}(U)^{\mathrm{lin.red.}}, \mathbb{G}_{m,k})\end{aligned}$$

Remark

1. S. Otabe proved the necessity of the aboved conjecture, so the question now is the converse.
2. S. Otabe showed that if G is a finite local solvable group scheme over k, then $G \in \pi_A^{\mathsf{L}}(U)$ iff there is an injection $\mathrm{Hom}_{\mathrm{grp.sch}}(G, \mathbb{G}_{m,k}) \hookrightarrow (\mathbb{Q}_p/\mathbb{Z}_p)^{\gamma+n-1}$.
3. If $U = \mathbb{A}_k^1$, then the conjecture just says that a finite local group scheme G is a quotient of $\pi^{\mathsf{L}}(U)$ iff G has no non-trivial characters. S. Otabe proved that all the Frobenius kernels of a semisimple simply connected algabraic group are in $\pi_A^{\mathsf{L}}(\mathbb{A}_k^1)$.

Theorem (S. Otabe, F. Tonini, Z)

Let k be an algebraically closed field of characteristic $p > 5$, and let G be a finite local non-abelian simple k-group scheme. Then $G \in \pi^{\mathsf{L}}_{A}(\mathbb{A}^1_k)$.


Thank you!